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We

Last time A k-linear map $\omega : \overbrace{V \times \dots \times V}^{k} \to \mathbb{R}$ is __alternating__ 19.1

if $\omega(v_1, \dots v_k) = -\omega(v_1 \dots \underset{i^{th}\text{ slot}}{v_j} \dots \underset{j^{th}\text{ slot}}{v_i} \dots v_k)$ for any $i<j$.

We proved:

18.2 A k-linear map $\omega : \overbrace{V \times \dots \times V}^{k} \to \mathbb{R}$ is alternating $\Leftrightarrow$ $\omega(v_1 \dots v_k) = 0$ for any $v_1 \dots v_k \in \mathbb{R}$ with $v_i = v_j$ for some $i<j$.

Remark 18.2 generalizes:

If $\omega : \overbrace{V \times \dots \times V}^{k} \to \mathbb{R}$ is k-linear and alternating and $v_1 \dots v_k \in V$ are linearly __dependent__ then $\omega(v_1 \dots v_k) = 0$.

Proof homework.

Our goal __Theorem__ (existence and uniqueness of determinants)

There exists a unique alternating n-linear map $\det : \mathbb{R}^n \times \dots \times \mathbb{R}^n = M_{n,n}(\mathbb{R}) \to \mathbb{R}$ such that $\det(e_1, \dots, e_n) = 1$ ($\{e_i\}_{i=1}^n$ standard basis of $\mathbb{R}^n$)

We defined $S_n = \{\sigma : \{1, \dots, n\} \to \{1, \dots n\} \mid \sigma \text{ is a bijection}\}$

S_n is a __group__ under composition.

Elements of S_n are called __permutations__

S_n is called __the group of permutations__ on n letters.

Lemma 19.1 Suppose $D : \mathbb{R}^n \times \dots \times \mathbb{R}^n = M_{n,n}(\mathbb{R}) \to \mathbb{R}$ is an alternating n-linear map. Then $\forall \ (a_{ij}) \in M_{n,n}(\mathbb{R})$

$$D((a_{ij})) = \sum_{\sigma \in S_n} a_{\sigma(1)1} \cdots a_{\sigma(n)n} D(e_{\sigma(1)}, \dots e_{\sigma(n)})$$

Proof The proof is "the same" as in the case $n=3$:

$$D\Big(\sum_{i_1} a_{i_1 1} e_{i_1}, \sum_{i_2} a_{i_2 2} e_{i_2}, \dots \sum a_{i_n n} e_{i_n}\Big) =$$

$$= \sum_{i_1, i_2, \dots i_n} a_{i_1 1} a_{i_2 2} \dots a_{i_n n} D(e_{i_1}, \dots e_{i_n})$$

If $i_1, \dots, i_n$ are __not__ all distinct $D(e_{i_1} \dots e_{i_n}) = 0$ by 18.2

If $(i_1, \ldots, i_n)$ are all distinct then $\sigma : \{1, \ldots n\} \to \{1, \ldots n\}$, $\sigma(j) = i_j$ is a bijection, hence an element of S_n and

$$(i_1, \ldots, i_n) = (\sigma(1), \ldots \sigma(n))$$

$$\Rightarrow D((a_{ij})) = \sum_{\sigma \in S_n} a_{\sigma(1)1} \cdots a_{\sigma(n)n} D(e_{\sigma(1)}, \ldots, e_{\sigma(n)}) \qquad \square$$

Note if $\sigma = id$, $D(e_{\sigma(1)}, \ldots e_{\sigma(n)}) = D(e_1, \ldots e_n) = 1$

if σ switches i and j, $i < j$

$$D(e_{\sigma(1)} \ldots e_{\sigma(n)}) = D(e_1, \ldots, \underset{i^{th}\text{ slot}}{e_j}, \ldots, e_i \ldots e_n)$$

$$= -D(e_1, \ldots e_n) = -1$$

We'll see shortly that $D(e_{\sigma(1)} \ldots e_{\sigma(n)}) = \pm 1$.
and the sign is uniquely determined by $\sigma \in S_n$.

Definition A permutation $\tau \in S_n$ is a transposition if it switches exactly two indices:

Ex $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ is a transposition, $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ is not a transposition

Notation For $1 \leq i < j \leq n$ $(i\,j) \in S_n$ is defined by

$$(i\,j)(k) = \begin{cases} k & k \neq i, j \\ j & k = i \\ i & k = j. \end{cases}$$

Ex $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1\,3)$

Lemma 19.2 Let $D : M_{n,n}(\mathbb{R}) \to \mathbb{R}$ be an alternating n linear map. Then

(a) $\forall \sigma \in S_n$ $\exists$ transpositions $\tau_1, \ldots, \tau_k$ s.t. $\sigma \tau_1 \tau_2 \ldots \tau_k = id$

(b) $D(e_{\sigma(1)}, \ldots e_{\sigma(n)}) = (-1)^k D(e_1 \ldots e_n) = (-1)^k$.

Proof Consider the ordered list $(\sigma(1), \sigma(2) \ldots \sigma(n))$

If $\sigma(1) = 1$ do noting.

If $\sigma(1) \neq 1$ $\exists j > 1$ s.t $\sigma(j) = 1$. Let $\tau_1 = (1\ j)$.

Then $(\sigma(\tau_1(1)), \sigma(\tau_1(2)), \dots \sigma(\tau_1(j)) \dots \sigma(\tau_1(n))) = (\overset{j\text{th slot}}{\sigma(j)}, \sigma(2), \dots, \sigma(1), \dots \sigma(n))$

$= (1, \sigma(2), \sigma(3), \dots, \sigma(1), \dots \sigma(n)) \Rightarrow \underbrace{D(e_1, e_{\sigma(2)}, \dots, \overset{j\text{th slot}}{e_{\sigma(1)}}, \dots e_{\sigma(n)})}_{D(e_{\sigma\tau_1(1)}, \dots e_{\sigma\tau_1(n)})} = (-1) D(e_{\sigma(1)} \dots e_{\sigma(n)})$

Continue.

If $e_{\sigma\tau_1(2)} = 2$ do nothing. Otherwise $\exists j > 2$ s.t $(\sigma\tau_1)(j) = 2$. Consider $\tau_2 = (2\ j)$. Then

$(\sigma\tau_1\tau_2(1), \sigma\tau_1\tau_2(2), \dots \sigma\tau_1\tau_2(n)) = (1, 2, \sigma\tau_1\tau_2(3), \dots \sigma\tau_1\tau_2(n))$

and

$$D(e_{\sigma(1)}, \dots e_{\sigma(n)}) = (-1) D(e_{\sigma\tau_1(1)}, \dots e_{\sigma\tau_1(n)})$$
$$= (-1)^2 D(e_{\sigma\tau_1\tau_2(1)}, e_{\sigma\tau_1\tau_2(2)}, \dots e_{\sigma\tau_1\tau_2(n)}).$$

In the end after $K \leq n$ steps we obtain K transpositions $\tau_K, \tau_{K-1}, \dots \tau_1 \in S_n$ s.t

(i) $\sigma\tau_1 \dots \tau_K = id$ and

(ii) $D(e_{\sigma(1)}, \dots, e_{\sigma(n)}) = (-1)^K D(e_1, \dots e_K) = (-1)^K$.

$\square$

Remark 19.3. For any transposition τ, $\tau = \tau^{-1}$.

So $\sigma\tau_1 \dots \tau_K = id. \Rightarrow \sigma = (\tau_1 \dots \tau_K)^{-1} = \tau_K^{-1}\tau_{K-1}^{-1} \dots \tau_1^{-1}$
$$= \tau_K\tau_{K-1} \dots \tau_1.$$

$\therefore \sigma = \tau_K\tau_{K-1} \dots \tau_1$

Remark 19.4 (i) We have proved that any permutation is a product of transpositions: $\forall \sigma \in S_n$ $\exists K \geq 1$ $\tau_1, \dots, \tau_K$ transpositions s.t $\sigma = \tau_1 \dots \tau_K$.

(ii) We also proved: if $D: M_{n,n}(\mathbb{R}) \to \mathbb{R}$ is n-lin and alternating and $\sigma = \tau_1 \dots \tau_K$ then $D(e_{\sigma(1)}, \dots e_{\sigma(n)}) = (-1)^K$.

So it feels like D is uniquely determined:

For any $\sigma \in S_n$ write $\sigma = \tau_1 \cdots \tau_k$; set $\epsilon(\sigma) = (-1)^k$

and then $(*)$ $D((a_{ij})) = \sum_{\sigma \in S_n} \epsilon(\sigma)\, a_{\sigma(1)1} \cdots a_{\sigma(n)n}$.

<u>Problem 1</u> Decomposition of a permutation σ into a product of transpositions is <u>not</u> unique

<u>Example</u> $(12)(13)(12)$: $\left\{ \begin{array}{l} 1 \to 2 \to 2 \to 1 \\ 2 \to 1 \to 3 \to 3 \\ 3 \to 3 \to 1 \to 2 \end{array} \right\} =$

So $(12)(13)(12) = (23)$. <u>Note</u> $(-1)^3 = (-1)^1$.

<u>Problem 2</u> Suppose we can show: there is a map
$\epsilon : S_n \to \{\pm 1\}$
so that for any $k \geq 1$ and any transpositions $\tau_1 \cdots \tau_k \in S_n$
$\epsilon(\tau_1 \cdots \tau_k) = (-1)^k$

Then any alternating n-linear map $D : M_{n,n}(\mathbb{R}) \to \mathbb{R}$ is given by $(*)$. $\Rightarrow$ we get uniqueness.

However, we still need to check: $(*)$ defines an alternation n-linear map.

<u>Note</u> The map $\overbrace{\mathbb{R} \times \cdots \times \mathbb{R}}^{n} \to \mathbb{R}$, $(x_1, \ldots x_n) \mapsto x_1 x_2 \ldots x_n$ is n-linear, so $\forall \sigma \in S_n$ $\forall (a_{ij}) \in M_{n,n}(\mathbb{R})$ the map $(a_{ij}) \mapsto a_{\sigma(1)1} \cdots a_{\sigma(n)n}$ is n-linear.

$\Rightarrow$

$\det((a_{ij})) := \sum_{\sigma \in S_n} \epsilon(\sigma)\, a_{\sigma(1)1} \cdots a_{\sigma(n)n}$ <u>is</u> n-linear

Why is det alternating?