02/27/2019
18.1
Wed.

Last time

Let $V_1 \ldots V_k$, U be vector spaces. A map
$\omega: V_1 \times \cdots \times V_k \to U$
is k-linear (bilinear if $k=2$, trilinear if $k=3$)
$\Leftrightarrow$ $\forall i$ ($1 \le i \le k$) $\forall \lambda_1, \lambda_2 \in \mathbb{R}$ $\forall w_1, w_2 \in V_i$, $\forall v_j \in V_j$, $j \ne i$,
$\omega(v_1, \ldots v_{i-1}, \lambda_1 w_1 + \lambda_2 w_2, v_{i+1}, \ldots v_k) =$
$\lambda_1 \omega(v_1 \ldots v_{i-1}, w_1, v_{i+1}, \ldots v_k) + \lambda_2 \omega(v_1 \ldots v_{i-1}, w_2, v_{i+1}, \ldots v_k)$.

Definition Let V, U be vector spaces. A k-linear map
$\omega: \underbrace{V \times \cdots \times V}_{k} \to U$ is alternating if
$\forall i, j$ with $1 \le i < j \le k$, $\forall v_1 \ldots v_k \in V$
$$\omega(v_1 \ldots v_{i-1}, v_i, v_{i+1} \ldots v_j \ldots v_k) = -\omega(v_1 \ldots v_{i-1}, v_j, v_{i+1} \ldots v_{j-1}, v_i, v_{j+1} \ldots)$$
switch

I.e. switching two variables in ω changes sign

Ex $D: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $D\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right) = ad - bc$
is alternating
$\omega: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ $\omega\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right) = ac - bd$
is not alternating.

Our goal is the following Theorem

Thm (existence and uniqueness of determinants)
For any $n \ge 1$ there exists a unique alternating n-linear map
$\det: \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n} \to \mathbb{R}$
such that $\det(e_1, e_2, \ldots e_n) = 1$ where $\{e_i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$.

Lemma 18.1 Let $B: V \times V \to \mathbb{R}$ be a bilinear map.
B is alternating (ie $B(v_1, v_2) = -B(v_2, v_1)$) $\Leftrightarrow$ $B(v, v) = 0$ $\forall v \in V$.

<u>Proof</u> $\forall\, v_1, v_2 \in V$

$$B(v_1+v_2, v_1+v_2) = B(v_1+v_2, v_1) + B(v_1+v_2, v_2) =$$
$$= B(v_1,v_1) + B(v_2,v_1) + B(v_1,v_2) + B(v_2,v_2).$$

$\Rightarrow$ <u>If</u> $B(u,u) = 0$ for all $u \in V$ then

$$0 = B(v_1+v_2, v_1+v_2) = 0 + B(v_2,v_1) + B(v_1,v_2) + 0$$
$$\Rightarrow B \text{ is alternating}$$

Conversely suppose $B(v_1,v_2) = -B(v_2,v_1)$ $\forall\, v_1, v_2 \in V$. Then

$$B(v,v) = -B(v,v) \qquad \forall\, v \in V$$
$$\Rightarrow 2B(v,v) = 0. \Rightarrow B(v,v) = 0.$$

$\square$

<u>Corollary 18.2</u> Suppose $\omega: \overbrace{V\times\cdots\times V}^{n} \to \mathbb{R}$ is an alternating n-linear map. Then $\omega(v_1, \ldots v_n) = 0$ whenever $\exists\, i,j$ s.t. $1 \le i < j \le n$ with $v_i = v_j$.

<u>Proof</u> Fix $v_1 - v_{i-1}, v_{i+1} -- v_{j-1}, v_{j+1} .. v_n \in V$ and consider

$$B(u,w) = \omega(v_1 .. v_{i-1}, u, v_{i+1}, - v_{j-1}, w, v_{j+1}, -- v_n).$$

Now apply 18.1.

$\square$

"<u>Example</u>" Suppose $D: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is an alternating bilinear map and $D\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right) = 1$. What's D?

$$D\left(\begin{pmatrix}a\\b\end{pmatrix}, \begin{pmatrix}c\\d\end{pmatrix}\right) = D(ae_1 + be_2, ce_1 + de_2) = ac\, D(e_1,e_1) + ad\, D(e_1,e_2)$$
$$+ bc\, D(e_2,e_1) + bd\, D(e_2,e_2) = ac\cdot 0 + ad\cdot 1 + bc\cdot(-1) + bd\cdot 0$$
$$= ad - bc$$

Success! We get the determinant of 2×2 matrices.

Can we do it for $n=3$?

Suppose $D: \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is (3-linear) alternating map with $D(e_1, e_2, e_3) = 1$. What is it?

$$D\left(\begin{pmatrix}a_{11}\\a_{21}\\a_{31}\end{pmatrix}, \begin{pmatrix}a_{12}\\a_{22}\\a_{32}\end{pmatrix}, \begin{pmatrix}a_{13}\\a_{23}\\a_{33}\end{pmatrix}\right) = D\left(\sum_{i_1=1}^{3} a_{i_1 1} e_{i_1}, \sum_{i_2=1}^{3} a_{i_2 2} e_{i_2}, \sum_{i_3=1}^{3} a_{i_3 3} e_{i_3}\right)$$

$$= \underbrace{\sum_{i_1,i_2,i_3=1}^{3} a_{i_1 1} a_{i_2 2} a_{i_3 3} D(e_{i_1}, e_{i_2}, e_{i_3})}_{9 \text{ terms}}$$

$$= \underbrace{\sum_{\substack{i_1,i_2,i_3\\ \text{distinct}}} a_{i_1 1} a_{i_2 2} a_{i_3 3} D(e_{i_1}\ e_{i_2}\ e_{i_3})}_{6 \text{ terms.}}$$

We need better bookkeeping if we want to understand n-linear alternating maps ...

<u>Definition</u> Fix $n \geq 1$. A <u>permutation</u> of the set $\{1, \ldots, n\}$ is a bijective map $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$.

We denote the set of all permutations of $\{1, \ldots, n\}$ by $\underline{S_n}$.

We can picture a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ by writing down <u>all</u> its values like this

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & & \sigma(n)\end{pmatrix} \quad \leftarrow \text{this is } \underline{\text{not}} \text{ a matrix.}$$

For example $S_1 = \{\binom{1}{1}\} = \{id_{\{1\}}\}$.

$S_2 = \{ id_{\{1,2\}}, \binom{1\ 2}{2\ 1}\}$

$S_3 = \{ id_{\{1,2,3\}}, \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}, \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}, \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}, \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}, \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$

<u>Lemma 18.3</u> $|S_n| = n!$

<u>Proof</u> Let $\sigma \in S_n$ be a permutation.

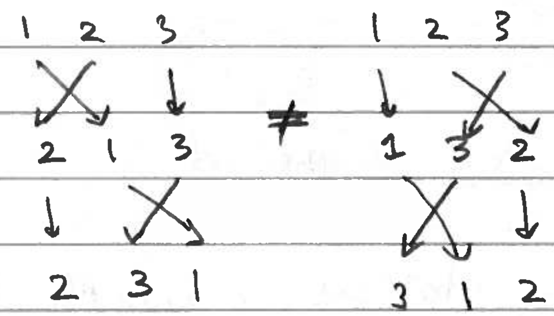
There are n choices of $\sigma(1)$, $n-1$ choices of $\sigma(2)$, $n-2$ choices of $\sigma(3)$ ... 1 choice of $\sigma(n)$.

$\Rightarrow |S_n| = n \cdot (n-1)(n-2) \cdots 2 \cdot 1 = n!$ $\square$

<u>Remark</u> For each $n \geq 1$, S_n is a <u>group</u>. This means:

1) we have a binary operation $\circ$ = composition
$$\circ : S_n \times S_n \longrightarrow S_n$$
2) $\circ$ is associative: $\forall\, \sigma, \tau, \mu \in S_n$, $(\sigma \circ \tau)\circ \mu = \sigma \circ (\tau \circ \mu)$
3) there is $id \in S_n$ s.t. $\sigma \circ id = \sigma = id \circ \sigma \quad \forall\, \sigma \in S_n$.
4) every element of S_n is invertible: $\forall \sigma \in S_n\ \exists\, \sigma^{-1} \in S_n$ s.t. $\sigma \circ \sigma^{-1} = id = \sigma^{-1} \circ \sigma$.

<u>Note</u> $\circ$ is <u>not</u> commutative in general:



Why do we care about S_n? Back to $D: \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

i_1, i_2, i_3 distinct $\Leftrightarrow$ $\exists\, \sigma \in S_3$ s.t. $i_1 = \sigma(1)$, $i_2 = \sigma(2)$, $i_3 = \sigma(3)$.

So we can rewrite the formula for D as
$$D\left(\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31}\end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32}\end{pmatrix}, \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33}\end{pmatrix}\right) = \sum_{\sigma \in S_3} a_{\sigma(1)1}\, a_{\sigma(2)2}\, a_{\sigma(3)3}\, D(e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)})$$

<u>Claim</u> $D(e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}) = \pm D(e_1, e_2, e_3). = \pm 1$

<u>Reason</u> Suppose $\sigma \in S_3$

if $\sigma = id$, $D(e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}) = D(e_1, e_2, e_3) = 1$

if $\sigma \neq id$ then either σ switches just 2 indices,

in which case $D(e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}) = -D(e_1, e_2, e_3)$

Or σ is a product of two permutations that only switch 2 indices (see Note above)

In these cases $D(e_{\sigma(1)}, e_{\sigma(2)}, e_{\sigma(3)}) = (-1)^2 D(e_1, e_2, e_3) = +1$.

<u>Moral</u>:

To construct determinants for all $n \geq 1$ we need to consistently assign ± 1 to permutations.