Last time The dual space of a vector space V is

$$V^* = \{ \ell : V \to \mathbb{R} \mid \ell \text{ linear} \} \equiv \mathcal{L}(V, \mathbb{R}) \equiv \mathrm{Hom}(V, \mathbb{R}).$$

If $\mathcal{B} = \{b_1, \ldots, b_n\}$ is a basis of a vector space V then $\mathcal{B}^* = \{b_1^*, \ldots, b_n^*\}$, where $b_i^* : V \to \mathbb{R}$ is defined by

$$b_i^*(b_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j, \end{cases}$$

is a basis of V^*

Hence if $\dim V = n < \infty$, $\dim V^* = n$.

(Note: $\dim W = n \iff \exists$ an isomorphism $\Psi : \mathbb{R}^n \to W$

So $\dim V = n \Rightarrow V$ and V^* are isomorphic.

Concretely $f : V \to V^*$ $f(\sum c_i b_i) = \sum c_i b_i^*$ is an isomorphism)

The transpose of a matrix A is the matrix A^T with

$$(A^T)_{ij} = A_{ji} \quad \text{for all indices } i, j.$$

Exercise $(AB)^T = B^T A^T$

Solution $((AB)^T)_{ij} = (AB)_{ji} = \sum_k A_{jk} B_{ki} = \sum_k (A^T)_{kj} (B^T)_{ik}$

$= \sum_k (B^T)_{ik} (A^T)_{kj} = (B^T A^T)_{ij}$ $\square$

Given a linear map $T : V \to W$ we have a linear map

$$T^* : W^* \to V^* \quad (\text{the book writes } T^t)$$

$T^*(\ell) := \ell \circ T$ for all $\ell \in W^*$.

We proved:

Lemma Let $\mathcal{A}$ be a basis of V, $\mathcal{A}^*$ dual basis of V^*, $\mathcal{B}$ a basis of W, $\mathcal{B}^*$ dual basis of W^* and $T : V \to W$ a linear map. Then

$$[T^*]_{\mathcal{A}^* \mathcal{B}^*} = ([T]_{\mathcal{B}\mathcal{A}})^T$$

Double duals

Given a vector space V we have a dual V^*. Since V^* is a vector space it has a dual $(V^*)^* =: V^{**}$.

Lemma 17.1 Suppose V is a finite dimensional vector space. Then the map $ev: V \to (V^*)^*$ defined by
$$(ev(v))(\ell) = \ell(v) \quad \text{for all } v \in V,\ \ell \in V^*$$
is a (natural) isomorphism.

Proof Step 1 For each $v \in V$ we have a linear map
$$ev(v): V^* \to \mathbb{R},\quad (ev(v))(\ell) = \ell(v).$$
Check $\forall \lambda_1, \lambda_2 \in \mathbb{R}$, $\ell_1, \ell_2 \in V^*$
$$ev(v)(\lambda_1 \ell_1 + \lambda_2 \ell_2) = (\lambda_1 \ell_1 + \lambda_2 \ell_2)(v) = \lambda_1 \ell_1(v) + \lambda_2 \ell_2(v)$$
$$= \lambda_1 \cdot (ev(v)\ell_1) + \lambda_2 (ev(v)\ell_2)$$

Step 2 The map
$$ev: V \to V^{**},\quad v \mapsto ev(v)$$
is linear

Check $\forall \lambda_1, \lambda_2 \in \mathbb{R}$, $v_1, v_2 \in V$, $\forall \ell \in V^*$
$$ev(\lambda_1 v_1 + \lambda_2 v_2)\ell = \ell(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \ell(v_1) + \lambda_2 \ell(v_2)$$
$$= \lambda_1 (ev(v_1)(\ell)) + \lambda_2 (ev(v_2)(\ell))$$
$$= (\lambda_1 ev(v_1) + \lambda_2 ev(v_2))(\ell)$$
Since $\ell \in V^*$ is arbitrary
$$ev(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 ev(v_1) + \lambda_2 ev(v_2).$$

Step 3
$$ev: V \to (V^*)^* \text{ is injective.}$$
Check suppose $ev(v) = 0$. Fix a basis $\mathcal{B} = \{b_1, \ldots, b_n\}$ of V. Then $v = \sum c_i b_i$ for some $c_1, \ldots, c_n \in \mathbb{R}$.

if $ev(v) = 0$, then $(ev(v))(\ell) = 0 \ \forall \ \ell \in V^*$.

In particular $0 = (ev(v))(b_i^*) = b_i^*(v) = b_i^*(\sum c_j b_j)$

$= \sum_j c_j b_i^*(b_j) = c_i.$

$\therefore c_i = 0 \;\forall i. \Rightarrow v = \sum c_i b_i = 0.$

$\Rightarrow N(ev) = \{0\}. \Rightarrow ev$ is injective.

Finally, $\dim V = n \Rightarrow \dim V^* = n \Rightarrow \dim (V^*)^* = n.$

$\Rightarrow ev: V \to (V^*)^*$ is an isomorphism (since it's injective). $\square$

Recall If $\{b_1 \ldots b_n\}$ is a basis of a vector space V then $\forall v \in V$ $\exists$ unique scalars $c_1 \ldots c_n \in V$ s.t

$$v = \sum c_i b_i.$$

The scalars $c_1, \ldots c_n$ depend on v; they are <u>functions</u> of v: $c_i = c_i(v)$. and $v = \sum c_i(v) b_i$

<u>Q.</u> What are these functions $c_i(v)$?

<u>A.</u> if $v = b_1$, $v = 1 \cdot b_1 + 0 \cdot b_2 + \ldots + 0 \cdot b_n$

if $v = b_2$ $v = 0 \cdot b_1 + 1 \cdot b_2 + 0 \cdot b_3 + \ldots + 0 \cdot b_n$ ..

if $v = b_j$ $v = 0 \cdot b_1 \ldots 0 \cdot b_{j-1} + 1 \cdot b_j + \ldots + 0 \cdot b_n$

$$\Rightarrow c_i(b_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j. \end{cases}$$

We guess: $c_i(v) = b_i^*(v).$

Check $b_i^*(v) = b_i^*\left(\sum_j c_j b_j\right) = \sum_j c_j b_i^*(b_j) = \sum c_j \delta_{ij} = c_i.$

We have proved: **Lemma 17.2** Let $B = \{b_1 \ldots b_n\}$ be a basis of V. Then $\forall v \in V$

$$v = \sum b_i^*(v) b_i$$ where $\{b_1^* \ldots b_n^*\}$ is the dual basis.

Remark In lecture 10 we saw that a choice of a basis $B = \{b_1 \ldots b_n\}$ of a vector space V defines an iso

$$\psi: \mathbb{R}^n \to V, \quad \psi\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \sum c_i b_i.$$

ψ has an inverse $\Phi: V \to \mathbb{R}^n$, $\Phi(v) = [v]_B$

$\phi\left(\sum c_i b_i\right) = \left[\sum c_i b_i\right]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$

Lemma 17.2 says: $\quad [v]_{\mathcal{B}} = \begin{pmatrix} b_1^*(v) \\ \vdots \\ b_n^*(v) \end{pmatrix}.$

<u>Multilinear maps</u> (see section 4.5 of text for a special case)

<u>Definition</u> Let $V_1, \dots, V_k$ be vector spaces. A function

$$\omega : V_1 \times \cdots \times V_k \to \mathbb{R}$$

is <u>k-linear</u> if it's linear in each variable: $\forall\, i \in \{1, \dots, k\}$, $\forall\, \lambda, \mu \in \mathbb{R}$, $v_1 \in V_1, \dots, v_{i-1} \in V_{i-1}$, $u, w \in V_i$, $v_{i+1} \in V_{i+1} \dots$

$$\omega(v_1, \dots v_{i-1}, \lambda u + \mu w, v_{i+1}, \dots v_k) = \lambda\, \omega(v_1 \dots v_{i-1}, u, v_{i+1}, \dots, v_k) + \mu\, \omega(v_1, \dots v_{i-1}, w, v_{i+1}, \dots v_k)$$

If $k=2$ we say ω is <u>bilinear</u>.

<u>Ex</u> The standard inner product $(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $(x, y) := \sum_{i=1}^{n} x_i y_i$ is bilinear.

<u>Ex</u> For any vector space V, the map $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$ $\quad \langle l, v \rangle = l(v)$ is <u>bilinear</u>.

<u>Ex</u> $D : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ $\quad D\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right) = ad - bc$ is bilinear.

<u>Ex</u> $D : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

$D\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & \cdots & \end{pmatrix} = \det\begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & \cdots & \cdots \\ a_{31} & \cdots & \end{pmatrix}$ is 3-linear.

<u>Remark</u> Multilinear maps are sometimes called <u>tensors</u>.