Last time:

- W subspace of a vector space V. Then $\pi: V \to V/W$, $\pi(v) = v + W$ is linear and surjective. $N(\pi) = W$, so π is not injective unless $W = \{0\}$

(15.1) If $\dim V < \infty$, $\dim V/W < \infty$ and $\dim V/W = \dim V - \dim W$

(15.2) 1st isomorphism theorem:

Suppose $T: V \to W$ is a linear map. Then
$$\bar{T}: V/N(T) \to W, \quad \bar{T}(v + N(T)) := T(v)$$
is a well-defined injective linear map with $R(\bar{T}) = R(T)$

Hence $\bar{T}: V/N(T) \to R(T)$ is an isomorphism.

- if $\dim V < \infty$, $\dim R(T) < \infty$ and
$$\dim R(T) = \dim (V/N(T)) = \dim V - \dim N(T).$$

(so 15.1 + 15.2 $\Rightarrow$ rank/nullity Theorem)

We haven't quite finished proving:

Cor 15.4 (second isomorphism theorem)

V a vector space, $W_1, W_2 \subseteq V$ two subspaces. Then $W_1 / W_1 \cap W_2$ is isomorphic to $(W_1 + W_2)/W_2$.

Hence, if $\dim W_1, \dim W_2 < \infty$
$$\dim(W_1 + W_2) - \dim W_2 = \dim W_1 - \dim(W_1 \cap W_2)$$
ie.
$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$

Proof The inclusion $i: W_1 \to W_1 + W_2$, $i(w) = w$ is linear.

$\pi: W_1 + W_2 \to (W_1 + W_2)/W_2$ is linear.

$\Rightarrow f = \pi \circ i: W_1 \to (W_1 + W_2)/W_2$ is linear.
$$f(w) = w + W_2$$

f is onto: given $v + W_2 \in (W_1 + W_2)/W_2$ $\exists\, w_1 \in W_1, w_2 \in W_2$ s.t $v = w_1 + w_2$. Since $(w_1 + w_2) - w_1 = w_2 \in W_2$,

$\Rightarrow \quad v + W_2 = (w_1 + w_2) + W_2 = w_1 + W_2 = f(w_1)$.

$\Rightarrow f$ is onto.

$N(f) = \{ w \in W_1 \mid w_1 + W_2 = W_2 \} = \{ w \in W_1 \mid w_1 \in W_2 \}$
$= W_1 \cap W_2$.

1st isomorphism theorem $\Rightarrow$

$$\bar{f} : W_1 / W_1 \cap W_2 \longrightarrow R(f) = (W_1 + W_2)/W_1$$
$$\bar{f}(w_1 + W_1 \cap W_2) = w_1 + W_2$$

is a well-defined isomorphism of vector spaces.

Dual vector spaces (sec 2.6 of text)

Definition Let V be a vector space. The dual vector space is the vector space

$$V^* := \{ l : V \to \mathbb{R} \mid l \text{ is linear} \}$$
$$\equiv \mathrm{Hom}(V, \mathbb{R}) \equiv \mathcal{L}(V, \mathbb{R}).$$

Lemma 16.1 (Thm 2.24 in text) Suppose $\{v_1, \dots v_n\}$ is a basis of a vector space V. For each $i = 1, 2, \dots n$ define

$$v_i^* : V \to \mathbb{R} \quad \text{by} \quad v_i^*(v_j) = \delta_{ij} := \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

Then $\{v_1^*, \dots v_n^*\}$ is a basis of V^*.

Moreover, $\forall\, l \in V^*$, $\quad l = \sum_{i=1}^n l(v_i)\, v_i^*$

Note $\{v_1^*, \dots, v_n^*\}$ is called a basis dual to the basis $\{v_1, \dots v_n\}$.

Proof (linear independence). Suppose $\exists c_1 \dots c_n \in \mathbb{R}$ s.t.

$$c_1 v_1^* + \dots + c_n v_n^* = 0 \leftarrow \text{zero-linear map!}$$

Then $\forall j \in \{1, \dots n\}$

$$0 = 0(v_j) = \left(\sum c_i v_i^*\right)(v_j) = \sum_{i=1}^n c_i\, v_i^*(v_j) =$$

$= c_1 v_1^*(v_j) + c_2 v_2^*(v_j) + \cdots + c_j v_j^*(v_j) + c_{j+1} v_{j+1}^*(v_j) + \cdots + c_n v_n^*(v_j)$

$= c_1 \cdot 0 + c_2 \cdot 0 \cdots + c_{j-1} \cdot 0 + c_j \cdot 1 + c_{j+1} \cdot 0 + \cdots + c_n \cdot 0 = c_j.$

$\Rightarrow$ $\{v_1^*, \ldots v_n^*\}$ is linearly independent.

To prove that $\quad \ell = \sum_{i=1}^n \ell(v_i) v_i^*$

it's enough to show:

$$(\star) \quad \ell(v_j) = \left(\sum_{i=1}^n \ell(v_i) v_i^*\right)(v_j) \quad \forall j.$$

Now

$$\left(\sum \ell(v_i) v_i^*\right)(v_j) = \ell(v_1) v_1^*(v_j) + \cdots + \ell(v_{j-1}) v_{j-1}^*(v_j) + $$
$$+ \ell(v_j) v_j^*(v_j) + \cdots + \ell(v_n) v_n^*(v_j)$$
$$= \ell(v_1) \cdot 0 + \cdots + \ell(v_{j-1}) \cdot 0 + \ell(v_j) \cdot 1 + \ell(v_{j+1}) \cdot 0 + \cdots$$
$$+ \ell(v_n) \cdot 0$$
$$= \ell(v_j).$$

$\Rightarrow$ $(\star)$ holds for all j. $\square$

Ex $\quad V = \mathbb{R}^n$, $\{e_1, \ldots e_n\}$ standard basis.

$\{e_1^*, \ldots e_n^*\}$ dual basis defined by

$$e_j^*(e_i) = \delta_{ij}$$

jth slot.

$\Rightarrow$ $e_j^* \leftrightarrow$ the row vector $(0 \cdots 0\ 1\ 0 \cdots 0)$

Dual maps

Suppose $T: V \to W$ is a linear map, $\ell \in W^*$.

Then we can compose: $V \xrightarrow{T} W \xrightarrow{\ell} \mathbb{R}$ and

$\ell \circ T \in V^*$.

Lemma 16.2 Let $T: V \to W$ be a linear map. Then

$$T^*: W^* \to V^*, \quad T^*(\ell) := \ell \circ T$$

is also linear.

Proof $\quad \forall \lambda_1, \lambda_2 \in \mathbb{R}, \quad \ell_1, \ell_2 \in W^*,$

$$T^*(\lambda_1 l_1 + \lambda_2 l_2) = (\lambda_1 l_1 + \lambda_2 l_2) \circ T$$
$$= \lambda_1 (l_1 \circ T) + \lambda_2 (l_2 \circ T)$$
$$= \lambda_1 T^*(l_1) + \lambda_2 T^*(l_2) \qquad \square$$

<u>Recall</u> If $A = \{a_1 \ldots a_n\}$ is a basis of V, $B = \{b_1 \ldots b_m\}$ a basis of W and $T: V \to W$ is linear, then

$$[T]_{BA} = (t_{ij}) \quad \text{where} \quad \boxed{T(a_j) = \sum_{i=1}^{m} t_{ij} b_i}$$

<u>Now:</u> $A \rightsquigarrow A^* = \{a_1^*, \ldots a_n^*\}$, $B \rightsquigarrow B^* = \{b_1^*, \ldots, b_m^*\}$

$T \rightsquigarrow T^*: W^* \to V^*$, $T^*(l) = l \circ T$.

So $\quad T^*(b_i^*) = \sum_{j=1}^{n} t_{ji}^* a_j^*$ for some $t_{ji}^* \in \mathbb{R}$.

What are they?

$$T^*(b_i^*) = \sum_{j=1}^{n} \left(T^*(b_i^*)(a_j)\right) a_j^*$$

$$\Rightarrow \boxed{t_{ji}^*} = \left(T^*(b_i^*)\right)(a_j) = (b_i^* \circ T)(a_j) = b_i^*\left(T(a_j)\right)$$
$$= b_i^*\left(\sum_{s=1}^{m} t_{sj} b_s\right)$$
$$= \sum_{s=1}^{m} t_{sj}\, b_i^*(b_s) = \sum_{s=1}^{m} t_{sj}\, \delta_{is} = \boxed{t_{ij}}$$

<u>Def</u> The transpose of an $m \times n$ matrix $D = (d_{ij})$ is the $n \times m$ matrix D^T with $(D^T)_{ij} = d_{ji}$

<u>Ex</u> $\quad D = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad D^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

<u>We proved</u> The matrix of T^* w.r.t the dual basis is the transpose of the matrix of T.