Last time: (1) $T: V \to W$ linear, $b \in W$

The set of solutions of $Tx = b$ is $v_0 + N(T)$ where v_0 is a particular solution of $Tx = b$.

(2) $W \subseteq V$ a subspace

$v \sim_W v' \iff v - v' \in W$ is an equivalence relation.

The equivalence class of v is

$$[v] = \{v' \in V \mid v' \sim v\} = v + W := \{v + w \mid w \in W\}$$

Recall For any equivalence relation $\sim$

$$[x] \cap [y] \neq \emptyset \iff [x] = [y]. \iff x \sim y$$

So $(v + W) \cap (v' + W) \neq \emptyset \iff v + W = v' + W \iff v - v' \in W.$

We proved: $\boxdot : \mathbb{R} \times V/W \to V/W$, $\lambda \boxdot (v + W) = (\lambda v) + W$

and $\boxplus : V/W \times V/W \to V/W$ $(v + W) \boxplus (u + W) = (v + u) + W$

are well-defined operations

They make V/W into a vector space.

Remark We have a map $\pi : V \to V/W$, $\pi(v) = v + W$.

$\boxdot$, $\boxplus$ are defined to make π linear: $\forall \lambda, \mu \in \mathbb{R}$, $u, v \in V$

$$(\lambda \boxdot \pi(v)) \boxplus (\mu \boxdot \pi(u)) = (\lambda \boxdot (v + W)) \boxplus (\mu \boxdot (u + W))$$
$$= (\lambda v + \mu u) + W = \pi(\lambda v + \mu u)$$

Note

$$N(\pi) = \{v \in V \mid \pi(v) = 0_{V/W}\} = \{v \in V \mid \pi(v) = 0 + W\}$$
$$= \{v \in V \mid v \sim_W 0\} = \{v \in V \mid v \in W\} = W.$$

$R(\pi) = V/W$ since $v + W = \pi(v)$ $\forall v + W \in V/W$.

Notation We now drop $\boxdot$ and $\boxplus$ and write

$\lambda(v + W)$ $(v + W) + (u + W)$ instead

<u>Lemma 15.1</u> Let V be a finite dimensional vector space, $W \subseteq V$ a subspace. Then V/W is finite dimensional and

$$\dim V/W = \dim V - \dim W.$$

<u>Proof 1</u> (using rank/nullity theorem)

Consider the linear map $\pi : V \to V/W$, $\pi(v) = v + W$

$N(\pi) = W$, $R(\pi) = V/W$. Rank/nullity thm $\Rightarrow$

$$\dim V/W = \dim R(\pi) = \dim V - \dim N(\pi) = \dim V - \dim W.$$

Proof 2 (direct)

Pick a basis $\{w_1 \ldots w_k\}$ of W; extend it to a basis $\{w_1, \ldots, w_k, z_1 \ldots z_\ell\}$ of V.

<u>Claim</u> $\{z_1 + W, \ldots z_\ell + W\}$ is a basis of V/W.

<u>Check</u> • $\forall\ v + W \in V/W$, $v = \sum \alpha_i w_i + \sum \beta_j z_j$ for some $\alpha_1 \ldots \alpha_k, \beta_1 \ldots \beta_\ell \in \mathbb{R}$ $\Rightarrow$

$$v + W = \left(\sum \alpha_i w_i + \sum \beta_j z_j\right) + W = \left(\left(\sum \alpha_i w_i\right) + W\right) + \sum_j \beta_j (z_j + W) = \beta_1(z_1 + W) + \ldots + \beta_\ell(z_\ell + W)$$

since $\sum \alpha_i w_i \in W$. $\Rightarrow \{z_1 + W, \ldots z_\ell + W\}$ spans V/W

• if $\exists\, c_1 \ldots c_\ell \in \mathbb{R}$ st.

$$\sum c_j (z_j + W) = 0 + W$$

Then $\sum c_j z_j \in W$

$\Rightarrow \exists\, \alpha_1 \ldots \alpha_k \in \mathbb{R}$ st $\sum c_j z_j = \sum \alpha_i w_i$.

Since $\{w_1 \ldots w_k, z_1 \ldots z_\ell\}$ is a basis of V, $c_1 = c_2 \ldots = 0$

$\therefore \{z_1 + W, \ldots z_\ell + W\}$ is lin. independent $\square$

<u>Theorem 15.2</u> (First isomorphism theorem) Let $T : V \to U$ be a linear map. The map

$$\overline{T} : V/N(T) \longrightarrow U, \quad \overline{T}(v + N(T)) = T(v)$$

is a well-defined injective linear map with $R(\overline{T}) = R(T)$

Consequently $\overline{T} : V/N(T) \longrightarrow R(T)$

$\overline{T}(v + N(T)) = T(v)$ is an isomorphism.

<u>Proof</u> To show that $\overline{T}$ is well-defined we need to check that if
$$v + N(T) = v' + N(T)$$
then $$T(v) = T(v').$$
Now $v + N(T) = v' + N(T) \Leftrightarrow v - v' \in N(T)$
$\Leftrightarrow v = v' + w$ for some $w \in N(T)$.
Hence $T(v) = T(v'+w) = T(v') + T(w) = T(v') + 0$ since $w \in N(T)$
$$= T(v').$$

Given $\lambda, \mu \in \mathbb{R}$, $v + N(T), v' + N(T) \in V/N(T)$
$$\overline{T}\left(\lambda(v+N(T)) + \mu(v'+N(T))\right)$$
$$= \overline{T}\left((\lambda v + \mu v') + N(T)\right)$$
$= T(\lambda v + \mu v')$ by definition of $\overline{T}$
$= \lambda T(v) + \mu T(v')$ since T is linear
$= \lambda \overline{T}(v + N(T)) + \mu \overline{T}(v' + N)$ (def of $\overline{T}$)

$N(\overline{T}) = \{v + N(T) \mid T(v) = 0\} = \{v + N(T) \mid v \in N(T)\}$
$$= \{0 + N(T)\} = \{0_{V/N(T)}\}$$
$\Rightarrow \overline{T}$ is 1-1.
It's easy to check that $R(\overline{T}) = R(T)$: $u \in R(T)$
$\Leftrightarrow u = T(v)$ for some $v \in V$
$\Leftrightarrow u = \overline{T}(v + N(T))$ for some $v + N(T) \in V/N(T)$.

$\therefore$ $\overline{T} : V/N(T) \longrightarrow R(T)$ is linear bijection, hence a linear isomorphism $\square$

<u>Corollary 15.3</u> (Rank/nullity Theorem)

Suppose $T : V \to W$ is a linear map and $\dim V < \infty$.

Then $\dim R(T) = \dim V - \dim N(T)$.

Proof By the 1st isomorphism theorem

$$\bar{T} : V/N(T) \longrightarrow R(T)$$

is an isomorphism. Since V is finite dimensional

$\dim V/N(T) < \infty$. $\Rightarrow$ $\dim R(T) < \infty$ and $\dim R(T) = \dim V/N(T)$

(isomorphisms take bases to bases).

Since $\dim V/N(T) = \dim V - \dim N(T)$

$\dim R(T) = \dim V - \dim N(T)$. $\square$

Corollary 15.4 Let V be a vector space, $W_1, W_2 \subseteq V$ two subspaces. Then $W_1/W_1 \cap W_2$ is isomorphic to $(W_1 + W_2)/W_2$.

Hence if $\dim V < \infty$

$$\dim W_1 - \dim(W_1 \cap W_2) = \dim(W_1 + W_2) - \dim W_2.$$

Proof

Consider the map $f : W_1 \longrightarrow (W_1 + W_2)/W_2$

$$f(w) = w + W_2$$

f is linear. f is onto: $u + W_2 \in (W_1 + W_2)/W_2$

is of the form $u = w_1 + w_2$ for some $w_1 \in W_1$, $w_2 \in W_2$.

$(w_1 + w_2) + W_2 = w_1 + W_2 = f(w_1)$ since $w_1 + w_2 - w_1 = w_2 \in W_2$.

$N(f) = \{ w \in W_1 \mid w + W_2 = 0 + W_2 \} = \{ w \in W_1 \mid w \in W_2 \}$

$= W_1 \cap W_2$

1st iso theorem: $\bar{f} : W_1/W_1 \cap W_2 \longrightarrow (W_1 + W_2)/W_2$

$$\bar{f}(w + (W_1 \cap W_2)) = w + W_2$$

is a well-defined linear isomorphism. $\square$