Last time:

- row echelon form, reduced row echelon form of a matrix A
- $\operatorname{rank}(A) = \operatorname{rank}((\text{reduced})\text{ row echelon form of } A)$
  $= \#$ of pivots in the row echelon form
- an "algorithm" for computing (reduced) row echelon form.

Remark Let A be an $m \times n$ matrix and suppose we want to solve $\boxed{Ax = b}$ where $b \in \mathbb{R}^m$.
We form the augmented matrix
$$(A|b)$$
Run the row echelon form algorithm on the augmented matrix $(A|b)$
We get $(QA|Qb)$
for some invertible $m \times m$ matrix Q (we don't actually compute Q, we compute QA and Qb).
Back substitute to solve $QAx = Qb$ (example below)
Any solution v of $QAx = Qb$ is also a solution of $Ax = b$ because $QAv = Qb \Rightarrow$
$Av = Q^{-1}QAv = Q^{-1}Qb = b$.

Example Suppose $(QA|Qb) = \left(\begin{array}{ccccc|c} 0 & \boxed{1} & 2 & 0 & 3 & 2 \\ 0 & 0 & 0 & \boxed{1} & -1 & 3 \end{array}\right)$

pivot columns

This is equivalent to
$$\begin{cases} 0x_1 + 1x_2 + 2x_3 + 0x_4 + 3x_5 = 2 \\ 0x_1 + 0x_2 + 0x_3 + 1x_4 - x_5 = 3 \end{cases}$$
x_1, x_3, x_5 are "free" variables. We can solve for x_2, x_4:
$$\begin{cases} x_2 = -2x_3 - 3x_5 + 2 \\ x_4 = x_5 + 3 \end{cases}$$
$\Rightarrow$ the general solution is of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 \\ -2x_2-3x_5+2 \\ x_3 \\ x_5+3 \\ x_5 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \\ 3 \\ 0 \end{pmatrix}$$

Note rank $QA = 2$. $\Rightarrow$ $\text{Null}(QA) = 5-2=3$

$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ is linearly independent, it lies in $N(QA)$

$\Rightarrow$ the set is a basis of $N(QA) = N(A)$.

Recall if $T: V \to W$ is linear, $b \in W$ and

$$T(v_1) = b = T(v_2)$$

Then $0 = T(v_1) - T(v_2) = T(v_1 - v_2) \Rightarrow v_1 - v_2 \in N(T)$.

Conversely, if $T(v) = b$ and $u \in N(T)$ Then

$$T(v+u) = T(v) + 0 = b.$$

So $v+u$ is also a solution of $Tx = b$

We conclude: if v_0 solves $Tx=b$, then the set of all solutions of $Tx=b$ is $\{v \mid T(v) = b\} = \{v_0 + u \mid u \in N(T)\} =: v_0 + N(T)$.

Quotient vector spaces

Definition Let V be a vector space and $W \leq V$ a subspace.

$\boxed{v_1 = v_2 \text{ mod } W}$ or $\boxed{v_1 \sim_W v_2}$ $\iff$ $v_1 - v_2 \in W$.

Lemma 14.1 The relation $\sim_W$ is an equivalence relation.

Proof (1) (reflexivity)

$v \sim v$ since $v - v = 0 \in W$

(2) (symmetry) if $v \sim v'$ then $v - v' = w$ for some $w \in W$

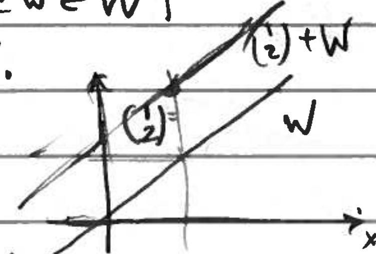
Since W is a subspace $W \ni -w = v' - v$. $\Rightarrow$ $v' \sim v$.

(3) transitivity Suppose $x, y, z \in V$, $x \sim y$ and $y \sim z$.

Then $x-y,\ y-z \in W$. Since W is a subspace
$W \ni (x-y)+(y-z) = x-y+y-z = x-z. \Rightarrow x \sim z.$

<u>Recall</u> If $\sim$ is an equivalence relation on a set X, $x \in X$ the <u>equivalence class</u> of x is the set
$$[x] := \{x' \in X \mid x' \sim x\}.$$

In our case $X = V$ and $\sim = \sim_W$. So for $v \in V$
$$[v] = \{v' \in V \mid v'-v \in W\}$$
$$= \{v' \in V \mid v'-v = w \text{ for some } w \in W\}$$
$$= \{v+w \mid w \in W\} := v+W.$$



<u>Ex</u> $V = \mathbb{R}^2$ $W = \operatorname{span}\binom{1}{1} = \mathbb{R}\binom{1}{1}$.
$$\left[\binom{1}{2}\right] = \binom{1}{2} + \mathbb{R}\binom{1}{1} = \left\{\binom{1}{2} + t\binom{1}{1} \,\middle|\, t \in \mathbb{R}\right\}.$$

<u>Notation</u> $V/W = \{v+W \mid v \in V\}$ the set of equivalence classes of the relation $\sim_W$.

<u>Lemma 14.2</u> Let W be a subspace of a vector space W, $\lambda \in \mathbb{R}$, $v_1, v_1', v_2, v_2' \in V$ with $v_1 \sim v_1'$, $v_2 \sim v_2'$. Then

(1) $\lambda v_1 \sim \lambda v_1'$

(2) $(v_1+v_2) \sim (v_1'+v_2')$.

<u>Proof</u> (1) $\lambda v_1 - \lambda v_1' = \lambda(v_1 - v_1') \in W$ since $v_1 - v_1' \in W$ and W is a subspace.

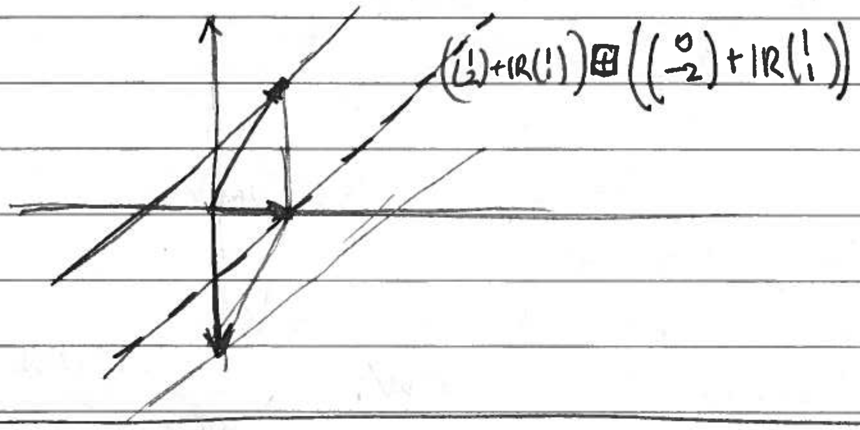
(2) $(v_1+v_2)-(v_1'+v_2') = (v_1-v_1') + (v_2-v_2') \in W$ since $v_1-v_1', v_2-v_2' \in W$ and W is a subspace. $\square$

<u>Consequence:</u> The maps $\boxdot : \mathbb{R} \times V/W \to V/W$ $\lambda \boxdot (v+W) := \lambda v + W$
and $\boxplus : V/W \times V/W \to V/W$, $(v_1+W) \boxplus (v_2+W) := (v_1+v_2)+W$

are well-defined.

<u>Example</u> $V = \mathbb{R}^2 \quad W = \mathbb{R}\begin{pmatrix}1\\1\end{pmatrix}$

$$\left(\begin{pmatrix}1\\2\end{pmatrix}+\mathbb{R}\begin{pmatrix}1\\1\end{pmatrix}\right) \boxplus \left(\begin{pmatrix}0\\-2\end{pmatrix}+\mathbb{R}\begin{pmatrix}1\\1\end{pmatrix}\right) = \left(\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}0\\-2\end{pmatrix}\right)+\mathbb{R}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}+\mathbb{R}\begin{pmatrix}1\\1\end{pmatrix}$$



<u>Lemma 14.3</u> Let W be a subspace of a vector space V. The set V/W of equivalence classes with the operations $\boxdot$, $\boxplus$ and $0_{V/W} := 0_V + W$ is a vector space.

<u>Sketch of proof</u> We need to check 8 properties (see lecture 2)

- $\boxplus$ is commutative: $(v+W) \boxplus (v'+W) = (v+v')+W = (v'+v)+W = (v'+W) \boxplus (v+W)$.
- $\forall\, v+W \in V/W$, $(v+W) \boxplus (0+W) = (v+0)+W = v+W$
- $\forall\, \lambda \in \mathbb{R}$, $\forall\, v+W,\ v'+W \in V/W$

$$\lambda \boxdot \left((v+W) \boxplus (v'+W)\right) = \lambda \boxdot \left((v+v')+W\right) = \lambda(v+v')+W =$$
$$= (\lambda v + \lambda v')+W = (\lambda v + W) \boxplus (\lambda v' + W)$$
$$= \left(\lambda \boxdot (v+W)\right) \boxplus \left(\lambda \boxdot (v'+W)\right)$$

+ 5 more checks...

<u>Next time</u>: • If V is finite dimensional $\dim V/W = \dim V - \dim W$

- $\pi : V \to V/W$, $\pi(v) = v+W$ is <u>linear</u> and $\operatorname{null}(\pi) = W$.