02/15/2019
Fr
13.1

Last time

- Defined elementary row operations on an $m \times n$ matrix A
- observed: an elementary row operation corresponds to multiplying A on the left by a matrix E.
  One obtains E by applying the corresponding operation to I_m.
- Elementary matrix E is invertible

$\Rightarrow$ - if $\exists$ a sequence of row operations transforming A into I_m, then we can compute A^{-1} by applying the same sequence to I_m.

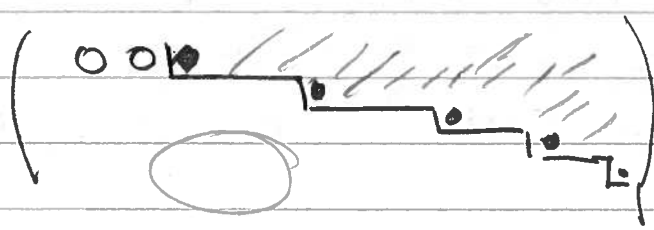
Today: row echelon form, reduced row echelon form and an algorithm for producing them. (Treil p40–54)

Def A matrix is in a row echelon form if it satisfies two conditions

1) All rows with all entries 0 (if any) are below all nonzero entries.

For a row with nonzero entries the leftmost nonzero entry is called the leading entry or a pivot.

2) For any nonzero row the pivot is strictly to the right of the pivot of the row above it



Ex
pivots.
$$\begin{pmatrix} 0 & \boxed{4} & 2 & 3 & 0 & 5 \\ 0 & 0 & \boxed{3} & 7 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

"Algorithm" for transforming a matrix into the row echelon form by elementary row operations. (Gaussian elimination)

Main step: • Find leftmost nonzero column of the matrix

• By switching rows, if necessary, arrange for the uppermost entry to be nonzero

• Use row operations to turn all the entries below that nonzero entry to 0.

"Algorithm": • Apply main step to an $m\times n$-matrix A

• Apply main step to the submatrix formed by rows $2, 3, \dots m$

• Apply main step to the submatrix formed by rows $3, \dots m$

Keep going untill you run out of nonzero rows.

Ex
$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 2 & 1 \\ 0 & 2 & 1 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2-6 & 1-9 \\ 0 & 0 & 1-4 & 2-6 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & -4 & -8 \\ 0 & 0 & -3 & -4 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & -4 & -8 \\ 0 & 0 & 0 & -4-3\cdot(\frac{-8}{-4}) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & -4 & -8 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Definition A matrix is in a reduced row echelon if

• it is in a row echelon form and furthermore

1) all pivots are 1

2) all entries above pivots (and below pivots) are 0.

It's easy to transform a matrix from a row echelon form into a reduced row echelon form:

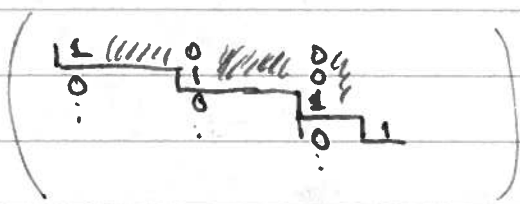
Ex
$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & -4 & -8 \\ 0 & 0 & 0 & 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Lemma 13.1 Suppose an $m \times n$ matrix A is in a reduced row echelon form. Then the columns with pivots form a basis of the range of A (ie. of the linear map defined by A).

Proof Range of A = span $\{$columns of $A\}$

It's easy to see that the set of pivot columns of A is linearly independent — it's a subset of the standard basis.

It's also easy to see that they span $\mathbb{R}^k \times \{0\} \subseteq \mathbb{R}^m$ where k = # of nonzero rows, and all the other columns of A <u>are</u> in this subspace $\mathbb{R}^k \times \{0\}$ of $\mathbb{R}^m$.



Lemma 13.2 (i) For any $m \times n$ matrix A and for any <u>invertible</u> $m \times m$ matrix Q, $\quad \operatorname{rank}(QA) = \operatorname{rank}(A)$

(while in general $\operatorname{range}(QA) \neq \operatorname{range}(A)$)

(ii) null space of QA = null space of A.

Consequently to compute rank (and nullity) of A, it's enough to row reduce A, and then count the pivots.

The proof of 13.2 (i) uses a lemma:

Lemma 13.3 Suppose $T: V \to W$ is a linear isomorphism, $U \subset V$ a subspace of dimension k. Then $T(U) \subseteq W$ is a subspace of dimension k as well.

Proof Pick a basis $\{b_1 \ldots b_k\}$ of U.

Then $\{T(b_1), \ldots T(b_k)\}$ spans $T(U)$: $w \in T(U)$ $\Rightarrow$ $w = T(u)$ for some $u \in U$. $\Rightarrow$ $u = \sum \lambda_i b_i$ for some $\lambda_1 \ldots \lambda_k$ $\Rightarrow$ $w = T(\sum \lambda_i b_i) = \sum \lambda_i T(b_i)$.

Also if $0 = \sum c_i T(b_i)$ for some $c_1 \ldots c_k$, then

$0 = T(\Sigma c_i b_i) \Rightarrow \Sigma c_i b_i = 0$ since T is 1-1

$\Rightarrow c_1 = c_2 = \ldots = c_k = 0$ since $\{b_1 \ldots b_k\}$ is lin indep.

Proof of 13.2 (i)

$\text{Rank}(A) = \dim(A(\mathbb{R}^n))$

$\text{Rank}(QA) = \dim(Q(A(\mathbb{R}^n)))$

Since Q is invertible, 13.3 $\Rightarrow \dim Q(A(\mathbb{R}^n)) = \dim A(\mathbb{R}^n)$.

(ii) $v \in N(A) \Leftrightarrow Av = 0 \Leftrightarrow Q(Av) = 0$ since Q is invertible

$\Leftrightarrow v \in N(QA)$

(iii) Let A' be a matrix in (reduced) row echelon form obtained from A. Then $\exists$ elementary matrices $E_1, \ldots E_\ell$ s.t. $A' = (E_\ell \ldots E_1)\, A$.

Let $Q = (E_\ell \ldots E_1)$ and apply (i), (ii)

We get $\text{rank}(A') = \text{rank}(A)$, $\text{nullity}(A') = \text{nullity}(A)$ $\square$

<u>Note:</u> if A is $m \times n$ and $\text{rank}\, A = k$

nullity of $A = n - k$, by rank/nullity theorem.

<u>Exercise</u> $T: V \to W$ linear, $Q: V \to V$ invertible.

Is it true in general that $\text{Null}(T) = \text{Null}(TQ)$?