Homework 5 posted.

02/13/2019
We

Last time

11.2 (Theorem 2.11 in text) Let $(V, \mathcal{A})$, $(W, \mathcal{B})$, $(Z, \mathcal{C})$ be three finite dimensional vector spaces with bases, $T: V \to W$, $U: W \to Z$ linear maps. Then

$$[U \circ T]_{\mathcal{C}\mathcal{A}} = [U]_{\mathcal{C}\mathcal{B}} \cdot [T]_{\mathcal{B}\mathcal{A}}.$$

"Application" $\mathcal{S} = \{e_1, \ldots, e_n\}$ standard basis of $\mathbb{R}^n$

$\mathcal{A} = \left\{ \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ \vdots \\ a_{n2} \end{pmatrix}, \ldots, \begin{pmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{pmatrix} \right\}$ another basis. Then

$$[id_{\mathbb{R}^n}]_{\mathcal{S}\mathcal{A}} = \left([a_1]_{\mathcal{S}} \,\middle|\, [a_2]_{\mathcal{S}} \,\middle|\, \ldots \,\middle|\, [a_n]_{\mathcal{S}} \right) = \left(\begin{matrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{matrix} \,\middle|\, \begin{matrix} a_{12} \\ \vdots \\ a_{n2} \end{matrix} \,\middle|\, \ldots \,\middle|\, \begin{matrix} a_{1n} \\ \vdots \\ a_{nn} \end{matrix} \right)$$

$$11.2 \Rightarrow \quad [id_{\mathbb{R}^n}]_{\mathcal{S}\mathcal{A}} \cdot [id_{\mathbb{R}^n}]_{\mathcal{A}\mathcal{S}} = [id_{\mathbb{R}^n}]_{\mathcal{S}\mathcal{S}} = I_n$$

$$\Rightarrow \quad [id_{\mathbb{R}^n}]_{\mathcal{A}\mathcal{S}} = \left([id_{\mathbb{R}^n}]_{\mathcal{S}\mathcal{A}}\right)^{-1} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & & a_{nn} \end{pmatrix}^{-1}.$$

Another "application": $\mathcal{A} = \{a_1, \ldots, a_n\}$, $\mathcal{B} = \{b_1, \ldots, b_n\}$ two bases of $\mathbb{R}^n$. What's $[id_{\mathbb{R}^n}]_{\mathcal{B}\mathcal{A}}$?

Solution:

$$[id]_{\mathcal{B}\mathcal{A}} = [id]_{\mathcal{B}\mathcal{S}} [id]_{\mathcal{S}\mathcal{A}} = \left([id]_{\mathcal{S}\mathcal{B}}\right)^{-1} [id]_{\mathcal{S}\mathcal{A}} = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & & b_{nn} \end{pmatrix}^{-1} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & & a_{nn} \end{pmatrix}$$

Elementary row operations and matrices (ch 3.1)

Def Let A be an $m \times n$ matrix. An <u>elementary row operation</u> on A is one of the following:

(1) interchange (switch) two rows of A

(2) multiply a row of A by a non zero number

(3) add a scalar multiple of one row to another row.

Theorem (Theorem 3.1 in text, p. 41 in Treil)

- An elementary row operation on a matrix A corresponds to multiplying A on the right by an elementary matrix E.
- E is obtained from I_m by the corresponding row operation.

<u>Proof</u> Suppose we want to interchange i^{th} and j^{th} row of A. Interchange i^{th} and j^{th} row of I_m. We get

$$E = \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccc} & & i & & j & \end{array} \\ \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & 1 & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & 1 & & & 0 \\ & & & & & & & 1 \\ & & & & & & & & \ddots & \\ & & & & & & & & & 1 \end{pmatrix} \end{array} . \qquad E \cdot k^{th} \text{ column of } A =$$

$$= \begin{pmatrix} & & \\ & 0 & 1 \\ & 1 & 0 \\ & & \end{pmatrix} \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{mk} \end{pmatrix} = \begin{array}{c} \begin{pmatrix} a_{1k} \\ \vdots \\ a_{i-1,k} \\ a_{j,k} \\ a_{i+1,k} \\ \vdots \\ a_{i,k} \\ \vdots \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \\ \leftarrow i^{th} \\ \\ \\ j^{th} \\ \\ \end{array}$$

Suppose we want to multiply i^{th} row of A by $c \neq 0$. The corresponding elementary matrix is

$$E = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & c & & \\ & & & & 1 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \; i^{th}$$

$$i^{th} \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & c & & \\ & & & & 1 & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{ik} \\ \vdots \\ a_{mk} \end{pmatrix} i^{th} = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{i-1k} \\ ca_{ik} \\ a_{i+1k} \\ \vdots \\ a_{mk} \end{pmatrix} \quad \checkmark$$

The last case is similar. $\square$

Ex $m=5$. We want to add $\frac{1}{3}\cdot 2^{nd}$ row to 4^{th} row

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \frac{1}{3}a_2 + a_4 \\ a_5 \end{pmatrix}$$

Remark. Since elementary row operations are reversible, the corresponding elementary matrices are invertible.

$$\begin{array}{c c} & \begin{matrix} \quad i & \quad \quad j \quad \end{matrix} \\ \begin{matrix} \\ i \\ j \\ \\ \end{matrix} & \begin{pmatrix} 1 & & & & & \\ & \ddots 1 & & & & \\ & & 0 & \cdots 1 & & \\ & & 1 & \ddots 0 & & \\ & & & & 1 & \\ & & & & & \ddots 1 \end{pmatrix}^{-1} \end{array} = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & \ddots \end{pmatrix}$$

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & c & \\ & & & \ddots \; 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & & & & \\ & \ddots 1 & & & \\ & & \frac{1}{c} & & \\ & & & 1 & \\ & & & & \ddots \; 1 \end{pmatrix} \quad \text{etc.}$$

Consequences 1) Suppose we can convert a matrix A into I_m after a sequence of elementary row operations. Then we can compute A^{-1}.

Reason $\exists$ $E_1, \ldots, E_\ell$, elementary matrices, s.t.

$$E_\ell \cdots E_1 A = I_m$$

$$\Rightarrow \quad A = E_1^{-1} E_2^{-1} \cdots E_\ell^{-1} \cdot I_m = (E_\ell \cdots E_1)^{-1}$$

$\Rightarrow$ A is invertible with the inverse $E_\ell \cdots E_1$.

2) We can compute A by applying row operations to A and I_m simultaneously. (see p161 of text)

Example:

$$\begin{pmatrix} 0 & 1 & 1 \\ 2 & 4 & 2 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = ?$$

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 2 & 4 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & -1 \\ 1 & 2 & 0 & 0 & \frac{1}{2} & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & -2 & \frac{1}{2} & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

<u>Check</u> $$\begin{pmatrix} -2 & \frac{1}{2} & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 2 & 4 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2+2 & -2+1+1 \\ 0 & 1 & 1-1 \\ 0 & 0 & 1 \end{pmatrix} \quad \checkmark$$