Last time:

9.1 A linear map $T: \mathbb{R}^n \to \mathbb{R}^m$ uniquely defines a matrix (T_{ij}) so that

$$T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (T_{ij})\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \forall \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n.$$

$$\begin{pmatrix} T_{1j} \\ \vdots \\ T_{mj} \end{pmatrix} := T(e_j) \leftarrow j^{th} \text{ column of the matrix } (T_{ij})$$

Ex $T: \mathbb{R}^2 \to \mathbb{R}^3$ $\quad T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 + x_2 \\ x_2 - x_1 \end{pmatrix}$

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \\ T_{31} & T_{32} \end{pmatrix} = \left(T\begin{pmatrix} 1 \\ 0 \end{pmatrix} \middle| T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 1+0 & 0+1 \\ 0-1 & 1-0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Check $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1\cdot x_1 + 0\cdot x_2 \\ 1\cdot x_1 + 1\cdot x_2 \\ (-1)x_1 + 1\cdot x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_2 - x_1 \end{pmatrix}$ ✓

Remark 10.1 Any $m \times n$ matrix $A = (a_{ij})$ defines a linear map

$$A: \mathbb{R}^n \to \mathbb{R}^m \qquad A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix}$$

Note The text denotes the linear map $\mathbb{R}^n \to \mathbb{R}^m$ corresponding to the matrix A by L_A. Thus

$$L_A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A\cdot\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

lin map. $\quad$ mult by a matrix

Easy to see $L_A(e_j) = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$

So The map $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \to M_{m\times n}(\mathbb{R})$

$$T \mapsto (T(e_1) | \cdots | T(e_n))$$

is a bijection.

Recall $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ = the set of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$.
$\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ is a vector space. The text calls it $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.
(Theorem 2.8 in text)

Lemma 10.3 The bijection $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \to M_{m,n}(\mathbb{R}^n)$ is linear: $\forall\ T, S \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ $\forall\ \lambda, \mu \in \mathbb{R}$

$$(\lambda T + \mu S)_{ij} = \lambda\, T_{ij} + \mu\, S_{ij} \quad \forall\, i,j.$$

Why is the lemma true? Two reasons:

1) Let V, W be two vector spaces, $v \in V$ a vector.
We then have a map "evaluation at v"

$$ev_v : \mathrm{Hom}(V, W) \to W, \quad ev_v(T) := T(v)$$

It's linear:

$$ev_v(\lambda T + \mu S) = (\lambda T + \mu S)(v) = \lambda\, T(v) + \mu\, S(v)$$
$$= \lambda\, ev_v(T) + \mu\, ev_v(S).$$

2) The map $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \to M_{m,n}(\mathbb{R})$ is

$$T \mapsto \big(ev_{e_1}(T) \,|\, ev_{e_2}(T) \,|\, \dots \,|\, ev_{e_n}(T)\big)$$

Ex $T, S \in \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$ $\quad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}, \quad S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}$

$$\big(T(e_1) \,|\, T(e_2)\big) = \left(T\begin{pmatrix} 1 \\ 0 \end{pmatrix} \,\middle|\, T\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\big(S(e_1) \,|\, S(e_2)\big) = \left(S\begin{pmatrix} 1 \\ 0 \end{pmatrix} \,\middle|\, S\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$\forall\ \lambda, \mu \in \mathbb{R}$

$$(\lambda T + \mu S)\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} y \\ x \end{pmatrix} + \mu\begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda y + \mu y \\ \lambda x + \mu \cdot 0 \end{pmatrix}$$

$$\lambda\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \mu\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda + \mu \\ \lambda & 0 \end{pmatrix}$$

and

$$\left((\lambda T + \mu S)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \,\middle|\, (\lambda T + \mu S)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & \lambda + \mu \\ \lambda & 0 \end{pmatrix} \quad \checkmark$$

We also proved

9.2 If $A: \mathbb{R}^n \to \mathbb{R}^m$, $B: \mathbb{R}^m \to \mathbb{R}^\ell$ linear, (A_{kj}), (B_{ik}) corr. matrices, Then

$$(B \circ A)_{ij} = \sum_{k=1}^{m} B_{ik} A_{kj}$$

Ex $\quad T: \mathbb{R}^2 \to \mathbb{R}^3 \qquad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x+y \\ y-x \end{pmatrix}$

$\quad S: \mathbb{R}^3 \to \mathbb{R}^2 \qquad S\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+3y-z \\ x-y \end{pmatrix}$

$$(T_{kj}) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad (S_{ik}) = \left(S\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \middle| S\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \middle| S\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 2 & 3 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$(S \circ T)\begin{pmatrix} x \\ y \end{pmatrix} = S\begin{pmatrix} x \\ x+y \\ y-x \end{pmatrix} = \begin{pmatrix} 2x + 3(x+y) - (y-x) \\ x - (x+y) \end{pmatrix} = \begin{pmatrix} 6x + 2y \\ -y \end{pmatrix}$$

$$\Rightarrow ((S \circ T)_{ij}) = \left((S \circ T)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \middle| (S \circ T)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 6 & 2 \\ 0 & -1 \end{pmatrix}$$

and

$$(S_{ik}) \cdot (T_{kj}) = \begin{pmatrix} 2 & 3 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2+3+1 & 3-1 \\ 1-1 & 1 \cdot 0 - 1 + 0 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 0 & -1 \end{pmatrix}$$

Bases, matrices and linear maps in general

(see Treil, p69)

Theorem 10.4 Let $T: V \to W$ be a linear map, $A = (a_1 \ldots a_n)$ a basis of V, $B = (b_1 \ldots b_m)$ a basis of W. There exists a unique $m \times n$ matrix $[T]_{BA}$ so that

$$[T]_{BA} [v]_A = [T(v)]_B \quad \text{for all } v \in V.$$

Recall $\Phi_A: V \to \mathbb{R}^n$, $\Phi_A(v) \to [v]_A$ is the linear isomorphism which is the inverse of $\Psi_A: \mathbb{R}^n \to V$, $\Psi_A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} x_i a_i$

(So $[\sum x_i a_i] = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \forall \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$).

Proof We have

$$\begin{tikzcd} V \arrow[r, "T"] & W \arrow[d, "\phi_B"] \\ \mathbb{R}^n \arrow[u, "\phi_A^{-1} = \psi_A"] & \mathbb{R}^m \end{tikzcd}$$

, 3 linear maps

The composite $\phi_B \circ T \circ \phi_A^{-1} : \mathbb{R}^n \to \mathbb{R}^m$ is linear, so corresponds to a unique matrix $[T]_{BA}$:

$$[T]_{BA} = \Big((\phi_B \circ T \circ \phi_A^{-1})(e_1) \Big| \cdots \Big| (\phi_B \circ T \circ \phi_A^{-1})(e_n) \Big)$$

$\forall j, \quad \phi_A^{-1}(e_j) = a_j. \Rightarrow (\phi_B \circ T \circ \phi_A^{-1})(e_j) = [T(a_j)]_B$ $\square$

$$\boxed{\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "{[\cdot]_A}"'] & W \arrow[d] \\ \mathbb{R}^n \arrow[r, dashed, "{[T]_{BA}}"'] & \mathbb{R}^m \end{tikzcd}}$$

Example $V = W = P_4$ - polynomials of degree ≤ 4.

$A = B = \{1, x, x^2, x^3, x^4\}$.

$T = \frac{d}{dx}$. $\quad [T]_{AA} = \left([\tfrac{d}{dx} 1]_A \,\middle|\, [\tfrac{d}{dx} x]_A \,\middle|\, [\tfrac{d}{dx} x^2]_A \,\middle|\, [\tfrac{d}{dx} x^3]_A \,\middle|\, [\tfrac{d}{dx} x^4]_A \right)$

$$= \left([0]_A \,\middle|\, [1]_A \,\middle|\, [2x]_A \,\middle|\, [3x^2]_A \,\middle|\, [4x^3]_A \right)$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Note $[Ta_j]_B = \begin{pmatrix} t_{1j} \\ \vdots \\ t_{mj} \end{pmatrix} \iff Ta_j = \sum_{i=1}^{m} t_{ij} b_i$

So $[T]_{BA} = (t_{ij})$ where $Ta_j = \sum_{i=1}^{m} t_{ij} b_i$