Systems of linear equations

According to wikipedea a <u>system of linear equations</u> is a collection of two or more linear equations involving the same set of variables. For example

$$(A)\quad \begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 3x_1 + 2x_2 + x_3 = 7 \\ 2x_1 + x_2 + 2x_3 = 1 \end{cases}$$

A general system of m linear equations with n unknowns looks like this

$$(**)\quad \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$

where $a_{11}, a_{12}, \ldots \ldots a_{mn}$ are numbers (rational, real, complex) and so are $b_1, \ldots b_n$.

How do we solve (**)? Eliminate variables by adding multiples of one row to other rows (or by switching rows). This changes the system but not the set of solutions

Back to example: (*). We change the system to

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ (3x_1 - 3\cdot x_1) + (2x_2 - 3\cdot 2x_2) + (x_3 - 3\cdot 3x_3) = 7 - 3\cdot 1 \\ (2x_1 - 2x_1) + (x_2 - 2\cdot 2x_2) + (2x_3 - 2\cdot 3x_3) = 1 - 2\cdot 1 \end{cases} \quad \left[\begin{matrix} \text{row}_2 - 3\,\text{row}_1 \\ \text{row}_3 - 2\,\text{row}_1 \end{matrix}\right]$$

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 0 - 4x_2 - 8x_3 = 4 \\ 0 - 3x_2 - 4x_3 = -1 \end{cases} \rightsquigarrow \begin{cases} x_2 + 2x_3 + 3x_3 = 1 \\ x_2 + 2x_3 = -1 \\ 3x_2 + 4x_3 = 1 \end{cases} \left[\begin{matrix} (-\tfrac{1}{4}\,\text{row}2) \\ (-1)\,\text{row}3 \end{matrix}\right]$$

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ x_2 + 2x_3 = -1 \\ (3x_2 - 3x_2) + (4x_3 - 3\cdot 2x_3) = 1 - 3(-1) \end{cases} \quad [\text{row}_3 - 3\cdot\text{row}_2] \longrightarrow$$

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ x_2 + 2x_3 = -1 \\ -2x_3 = 4 \end{cases} \longrightarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ x_2 + 2x_3 = -1 \\ x_3 = -2 \end{cases}$$

Now we are happy: plug $x_3 = -2$ into row_2 to get

$$x_2 - 4 = -1 \Rightarrow x_2 = 3$$

Now plug in $x_3 = -2$ $x_2 = 3$ into row_1:

$$x_1 + 2\cdot 3 + 3\cdot(-2) = 1$$

$$\Rightarrow x_1 = 1.$$

Note The variables x_1, x_2, x_3 are just place holders and we could have done everything above by manipulating the array of numbers:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 7 \\ 2 & 1 & 2 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 3-3\cdot 1 & 2-3\cdot 2 & 1-3\cdot 3 & 7-3\cdot 1 \\ 2-2\cdot 1 & 1-2\cdot 2 & 2-2\cdot 3 & 1-2\cdot 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -4 & -8 & 4 \\ 0 & -3 & -4 & -1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 3 & 4 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 3-3\cdot 1 & 4-3\cdot 2 & 1-3(\end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -2 & 4 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

Which converts to
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ x_2 + 2x_3 = -1 \\ x_3 = -1 \end{cases}$$

Now back-substitute

<u>Ex 1.2</u> Find all the solutions of the system

$$(3)\quad \begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 - x_2 + 2x_3 = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right) \qquad (\text{row}_2 - \text{row}_1)$$

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ \quad\; - x_2 + 2x_3 = 0 \end{cases} \Rightarrow \quad x_2 = 2x_3 \quad \text{and}$$

$$x_1 = -x_2 - x_3 = -2x_3 - x_3 = -3x_3$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3x_3 \\ 2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \quad \text{where } x_3 \text{ is } \underline{\underline{\text{arbitrary}}}$$

<u>Note!</u> system (3) has infinitely many solutions. Why?

Systems (**) can be written more compactly by introducing <u>matrices</u>

<u>Def</u> An $m \times n$ (real) matrix is an array of (real) numbers

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{m1} & \cdots & & a_{mn} \end{pmatrix} \equiv (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

We can think of a real $m \times n$ matrix as an element of $\mathbb{R}^{mn}$, but it's better to think of it as a function

$$\{1, \dots m\} \times \{1, \dots n\} \to \mathbb{R}, \quad (i,j) \longmapsto a_{ij}$$

Given an $m \times n$ matrix $A = (a_{ij})$ and $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

we can define the product

$$A\vec{x} = (a_{ij})\vec{x} \in \mathbb{R}$$

by taking dot product of rows of A and $\vec{x}$:

$$A\vec{x} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots \quad + a_{mn}x_n \end{pmatrix}$$

The system (AA) of linear equations the can be written compactly as

$$(\text{AAA}) \qquad A\vec{x} = \vec{b}$$

where $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\quad \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

It is useful to think of the $m \times n$ matrix A as defining a map (a function) $T_A : \mathbb{R}^n \to \mathbb{R}^m$

$$T_A(\vec{x}) = A\vec{x}$$

For example (AAA) has a solution $\iff$ $\vec{b}$ is in the image of the map $T_A : \mathbb{R}^n \to \mathbb{R}^m$.

More generally we'll show that equations (AAA) are very special: either the have no solutions, a unique solution or infinitely many solutions.

Compare with: $\quad ax^2 + bx + c = 0$

may have 0, 1, 2 (or infinitely many) solutions in $\mathbb{R}$ depending on what a, b, c are.

The map T_A is very special: it's linear!

That is, for any $\vec{x}, \vec{y} \in \mathbb{R}^n$ $\quad \lambda, \mu \in \mathbb{R}$

$$T_A(\lambda\vec{x} + \mu\vec{y}) = \lambda T_A \vec{x} + \mu T_A \vec{y}$$

(check that!)

Next time: we'll define vector spaces and linear maps between vector spaces.