Homework # 9, Math 416, **Prof. Eugene Lerman**
Due Wednesday, March 27, 2019 in class

1 A square matrix N is nilpotent if there is a natural number $k \geq 1$ so that $N^k = 0$.

(a) Prove that the determinant of a nilpotent matrix is zero.

(b) Is the converse true? Are there matrices with determinant 0 that are not nilpotent?

2 Suppose V is a 1-dimensional vector space and $T : V \to V$ is a linear map. Prove that there is a scalar λ (that depends on T) so that $T(v) = \lambda v$ for all $v \in V$ (same λ for all v).

3 Let B be an $n \times n$ matrix. Prove that the columns of B form a basis of $\mathbb{R}^n$ if and only if $\det B \neq 0$.

4 An $n \times n$ matrix A is orthogonal if and only if its transpose A^T is the inverse of A (so $AA^T = I = A^T A$). Prove that if A is orthogonal then $\det A = \pm 1$.

5

(a) Compute the determinant of the matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 5 \\ 1 & -3 & 0 \end{pmatrix}$

(b) Compute the determinant of the matrix $B = \begin{pmatrix} 4 & -6 & -4 & 4 \\ 2 & 1 & 0 & 0 \\ 0 & -3 & 1 & 3 \\ -2 & 2 & -3 & -5 \end{pmatrix}$

6 Prove that

$$\det \begin{pmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{pmatrix} = (y - x)(z - x)(z - y).$$

Hint: row reduction and properties of determinant.

7 A square matrix $A = (a_{ij})$ is lower triangular if all the entries above the diagonal are zero (i.e., $a_{ij} = 0$ if $i < j$). Prove that the determinant of a lower triangular matrix is the product of the diagonal entries: $\det A = \prod_{i=1}^{n} a_{ii}$. Hint: this should be quick.

8 Let A be an $n \times n$ matrix. What is $\det(-A)$ in terms of $\det A$? More generally what is $\det(\lambda A)$ in terms of $\lambda \in \mathbb{R}$ and $\det A$?