Homework # 8, Math 416, **Prof. Eugene Lerman**
Due Wednesday, March 13, 2019 in class

1 Let $b : V \times W \to \mathbb{R}$ be a bilinear map.

(a) Prove that for any $w \in W$ the function $w^{\#} : V \to \mathbb{R}$ defined by

$$w^{\#}(v) := b(v, w) \qquad \text{for all } v \in V \quad (\text{i.e., } w^{\#}(\cdot) = b(\cdot, w)\ )$$

is linear. That is, prove that $w^{\#}$ is in the dual vector space V^*.

(b) Prove that the map

$$\# : W \to V^*, \qquad w \mapsto w^{\#}$$

is linear.

(c) Prove that the null space $N(\#)$ of $\#$ is

$$\{w \in W \mid b(v, w) = 0 \text{ for all } v \in V\}.$$

(c) Now suppose that W is finite dimensional, $V = W^*$ and $b : W^* \times W \to \mathbb{R}$ is given by

$$b(\ell, w) := \ell(w).$$

Prove that in this case $\# : W \to (W^*)^*$ is injective.

2 Let V, W be two vector spaces. Prove that the space of bilinear maps

$$\text{Bilin}(V, W; \mathbb{R}) := \{b : V \times W \to \mathbb{R} \mid b \text{ is bilinear}\}$$

is a vector space. Here and elsewhere the scalar multiplication and addition of bilinear maps are defined by

$$(\lambda b)(v, w) := \lambda b(v, w) \text{ for all } \lambda \in \mathbb{R}, v \in V, w \in W$$

and

$$(b_1 + b_2)(v, w) := b_1(v, w) + b_2(v, w) \quad \text{for all } v \in V, w \in W.$$

3 Let V be a vector space, $\ell_1, \ell_2 : V \to \mathbb{R}$ two linear maps (that is, $\ell_1, \ell_2 \in V^*$).

(a) Prove that the map $b : V \times V \to \mathbb{R}$, $b(v_1, v_2) := \ell_1(v_1)\ell_2(v_2)$ is bilinear.

(b) Prove that $\ell_1 \wedge \ell_2 : V \times V \to \mathbb{R}$ defined by

$$(\ell_1 \wedge \ell_2)(v_1, v_2) := \ell_1(v_1)\ell_2(v_2) - \ell_1(v_2)\ell_2(v_1)$$

is bilinear and alternating.

4 Let V be a vector space. Suppose $\alpha : \overbrace{V \times \cdots \times V}^{k} \to \mathbb{R}$ is k-linear and alternating.

(a) Prove that for any $v_1, \ldots, v_{k-1} \in V$, $\lambda_1, \ldots, \lambda_{k-1} \in \mathbb{R}$

$$\alpha(v_1, \ldots, v_{k-1}, \textstyle\sum_{i=1}^{k-1} \lambda_i v_i) = 0.$$

(b) Prove that for any k linearly **dependent** vectors $v_1, \ldots, v_k \in V$

$$\alpha(v_1, \ldots, v_k) = 0.$$

(c) Prove that for any $v_1, \ldots, v_k \in V$, $\lambda \in \mathbb{R}$

$$\alpha(v_1 + \lambda v_2, v_2 \ldots, v_k) = \alpha(v_1, v_2, \ldots, v_k).$$

5 (a) Write the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$ as a product of transpositions.

(b) Compute the sign of the permutation σ.

6 Recall that an inversion of a permutation σ is a pair of indices i, j so that $i < j$ and $\sigma(i) > \sigma(j)$. The inversion number of σ is the number of all inversions of σ. What is the inversion number of $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$?

7 Let $T : V \to W$ be a linear map and $\alpha : \overbrace{W \times \cdots \times W}^{k} \to \mathbb{R}$ be k-linear and alternating. Define

$$T^*\alpha : \overbrace{V \times \cdots \times V}^{k} \to \mathbb{R}$$

by

$$(T^*\alpha)(v_1, \ldots, v_k) := \alpha(T(v_1), \ldots, T(v_k)) \quad \text{for all } v_1, \ldots, v_k \in V.$$

Prove that $T^*\alpha$ is k-linear and alternating.

8 Let V be a vector space. Let

$$\mathrm{Alt}^k(V) := \{\alpha : \overbrace{V \times \cdots \times V}^{k} \to \mathbb{R} \mid \alpha \text{ is } k\text{-linear and alternating}\}.$$

(a) Check that $\mathrm{Alt}^k(V)$ is a vector space

(b) Suppose V is 2-dimensional. Prove that $\dim \mathrm{Alt}^2(V) = 1$. Hint: pick a basis $\{b_1, b_2\}$ of V. Prove that any alternating bilinear map α is uniquely determined by $\alpha(b_1, b_2)$.

(c) Now suppose that $\dim V = 3$, $\{b_1, b_2, b_3\}$ is a basis of V and $\{b_1^*, b_2^*, b_3^*\}$ is the dual basis. Prove that $\{b_1^* \wedge b_2^*, b_1^* \wedge b_3^*, b_2^* \wedge b_3^*\}$ is a basis of $\mathrm{Alt}^2(V)$; the wedge of two linear functionals is defined in problem 3.