Homework # 7, Math 416, **Prof. Eugene Lerman**
Due Wednesday, March 6, 2019 in class

1
(a) Let $T : V \to W$ be a linear map, $U \subset V$ a subspace such that $U \subseteq N(T)$. Prove that

$$\bar{T} : V/U \to W$$

given by

$$\bar{T}(v + U) := T(v)$$

is a well-defined linear map.

(b) Let $\ell : V \to \mathbb{R}$ be a linear map and $U \subseteq V$ a subspace. Suppose $\ell(u) = 0$ for all $u \in U$. Prove that

$$\bar{\ell} : V/U \to \mathbb{R}, \qquad \bar{\ell}(v + U) = \ell(v)$$

is a well-defined linear map. Hint: Use part (a).

2 Suppose U is a subspace of a vector space V. The annihilator of U is the subset U° of the dual vector space V^* defined by

$$U^\circ := \{\ell \in V^* \mid \ell(u) = 0 \text{ for all } u \in U\}.$$

(a) Prove that U° is a subspace of V^*.

(b) Now assume that V is finite dimensional. Prove that $\dim U^\circ + \dim U = \dim V$. Hint: pick a basis of U, extend it to a basis of V. Now consider the dual basis of V^*...

3 Suppose $T : V \to W$ is a linear map between two finite dimensional vector spaces and $T^* : W^* \to V^*$ is the dual map (its "transpose").
(a) Prove that

$$N(T^*) = (R(T))^\circ.$$

(b) Prove that $\dim R(T^*) = \dim R(T)$. Hint: part (a), Problem 2(b), and rank/nullity theorem.

(c) Prove that for any $m \times n$ matrix A, the rank of A equals the rank of its transpose A^T.

4 Let $T : V \to W$ be a linear map between two finite dimensional vector spaces and $T^* : W^* \to V^*$ is the dual map (its "transpose"). Prove the T is onto if and only if T^* is 1-1. Hint: Problem 3(a).