Homework # 6, Math 416, **Prof. Eugene Lerman**
Due Wednesday, February 27, 2019 in class

1 Let $T : V \to W$ be a linear map. The level set of T at $b \in W$ is the set

$$T^{-1}(b) := \{v \in V \mid T(v) = b\}.$$

(a) Fix $b \in W$. Suppose $T^{-1}(b) \neq \varnothing$. Then there exists $v_0 \in V$ with $T(v_0) = b$. Prove that

$$T^{-1}(b) = v_0 + N(T),$$

where, as usual, $N(T)$ denotes the null space of T.

(b) By the first isomorphism theorem (lecture 15) $\bar{T} : V/N(T) \to W$ is an injective linear map. So for any b in the range of T (which is also the range of $\bar{T}$) the preimage $\bar{T}^{-1}(b)$ should be a one element set. Say $b = T(v_0)$. What is the coset $\bar{T}^{-1}(b)$?

2 Let W be a subspace of a vector space V. Prove that the operation $\boxplus : V/W \times V/W \to V/W$ (which is given by $(v + W) \boxplus (v' + W) := (v + v') + W$) is associative.
Hint: mimic the proof of commutativity of $\boxplus$ in Lecture 14.

3 Let W be a subspace of a vector space V.

(a) Given two equivalence classes $X, Y \in V/W$ define $X \maltese Y$ by

$$X \maltese Y := \{x + y \mid x \in X, y \in Y\} \tag{1}$$

Prove that for any $v, u \in V$

$$(v + W) \maltese (u + W) = (v + u) + W.$$

(b) Show that $\maltese$ is a well-defined binary operation on the elements of V/W and prove that

$$X \maltese W = X$$

for all $X \in V/W$.

4 Let V be a vector space. Then V is a subspace of V. Show that the quotient vector space V/V has exactly one element.

5 Let V be a vector space. Then $W = \{0\}$ is a subspace of V. Prove that the quotient vector space V/W is isomorphic to V.
Hint: Prove that in this case $\pi : V \to V/W$, $\pi(v) = v + W$ is an isomorphism.

6 Let V be the space of all continuous real valued functions on the real line: $V = C^0(\mathbb{R})$. Let

$$W = \{f \in C^0(\mathbb{R}) \mid f(0) = 0\}.$$

Prove that W is a subspace of V and that the quotient V/W is isomorphic to $\mathbb{R}$. Hint: consider the map $T : V \to \mathbb{R}$, $T(f) := f(0)$. Prove that T is a surjective linear map. What is its null space?

7 Let $T : V \to W$ be a linear map and $U \subseteq V$ a subspace.

(a) Prove that $T(U)$ is a subspace of W.

(b) Prove that $T(U)$ is isomorphic to $U/(U \cap N(T))$.

8 Let $T : V \to V$ be a linear map from a vector space V to itself. Define the powers T^k of T recursively by

$$T^0 = \mathrm{id}_V \qquad T^{k+1} := T \circ T^k \quad \text{for all } k \in \mathbb{N}.$$

(a) Prove that $N(T^k) \subseteq N(T^{k+1})$ and $R(T^{k+1}) \subseteq R(T^k)$ for all $k \in \mathbb{N}$.

(b) Prove that if V is finite dimensional then there is a $k \in \mathbb{N}$ so that $R(T^k) = R(T^{k+1})$.

(c) Prove that if $R(T^k) = R(T^{k+1})$ for some $k \in \mathbb{N}$ then $R(T^{k+s}) = R(T^k)$ for all $s > 0$.