Homework # 5, Math 416, **Prof. Eugene Lerman**
(corrected 2/17) **Due Wednesday, February 20, 2019 in class**

1 A row vector $(a_1, a_2, \dots, a_m)$ "is" an $1 \times m$ matrix and so should correspond to a linear map $A : \mathbb{R}^m \to \mathbb{R}$. What is $A(x)$ for $x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in \mathbb{R}^m$?

2 A column vector $\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{R}^n$ "is" an $n \times 1$ matrix and so corresponds to a linear map $B : \mathbb{R} \to \mathbb{R}^n$. What is $B(y)$ for $y \in \mathbb{R}$?

3 Suppose V is a finite dimensional vector space, W_1, W_2 two subspaces. Prove that

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$

Hint: see Lecture 6.

4 Let V be a finite dimensional vector space and $P : V \to V$ a linear map such that $P \circ P = P$. Prove that V is the direct sum of the null space $N(P)$ and the range $R(P)$ of P:

$$V = N(P) \oplus R(P)$$

Hints: (i) Show that $N(P) \cap R(P) = \{0\}$
(ii) Use problem 3 above to show that $\dim(N(P) + R(P)) = \dim(N(P)) + \dim(R(P))$.
(iii) Use (ii) and rank/nullity to argue that $N(P) + R(P) = V$.

5 Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 7 & 3 \\ 2 & 3 & 4 \end{pmatrix}$$

by augmenting it with the identity matrix I_3 and performing elementary row operations (see an example on p. 53 of Teil's book).

6 Consider the linear map $T = \frac{d}{dx}$ from the space $\mathcal{P}_1$ of polynomials of degree ≤ 1 to itself. Consider the two bases $\mathcal{B} = \{1, x\}$ and $\mathcal{B}' = \{1 + x, 1 - x\}$ of $\mathcal{P}_1$. Compute the matrix $[T]_{\mathcal{B}'\mathcal{B}}$.

Hints: (i) $[T]_{\mathcal{B}'\mathcal{B}} = [\mathrm{id}_{\mathcal{P}_1}]_{\mathcal{B}'\mathcal{B}}[T]_{\mathcal{B}\mathcal{B}}$.
(ii) $[id]_{\mathcal{B}'\mathcal{B}} = ([id]_{\mathcal{B}\mathcal{B}'})^{-1}$. Why?

7 Recall that a linear map is uniquely determined by its values on a basis. Let $\mathcal{B} = \{b_1, \dots, b_n\}$ be a basis of a vector space V. For each index $i \in \{1, \dots, n\}$ there is a unique linear map $\ell_i : V \to \mathbb{R}$ with

$$\ell_i(b_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Prove that $\{\ell_1, \dots, \ell_n\}$ is a basis of the space $\mathrm{Hom}(V, \mathbb{R})$ of all linear maps from V to $\mathbb{R}$.
Hints: (i) Given $T \in \mathrm{Hom}(V, \mathbb{R})$ compare the values of T on the basis vectors with the values of the linear map $\sum_{i=1}^n T(b_i)\ell_i$.
(ii) Suppose $\sum c_i \ell_i = 0$. Evaluate the sum on a basis vector b_j. What do you get?

8 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map, $\mathcal{B}$ a basis of $\mathbb{R}^n$ and $\mathcal{S}$ the standard basis of $\mathbb{R}^n$.
(a) Prove that $[T]_{\mathcal{B}\mathcal{B}} = ([\mathrm{id}]_{\mathcal{S}\mathcal{B}})^{-1} [T]_{\mathcal{S}\mathcal{S}}[\mathrm{id}]_{\mathcal{S}\mathcal{B}}$.
(b) Compute $[T]_{\mathcal{B}\mathcal{B}}$ where $n = 2$, $\mathcal{B} = \{(1, 1)^T, (1, 2)^T\}$ and

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + y \\ x - 2y \end{pmatrix}$$