Homework #4, Math 416, **Prof. Eugene Lerman**
Due Wednesday, February 13, 2019 in class

1 Let W be a vector space, $w_1, \dots, w_n \in W$ a collection of vectors (not necessarily distinct). Prove that the map $\Psi : \mathbb{R}^n \to W$ defined by

$$\Psi \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} x_i w_i$$

is linear.

2 Consider a general 3×4 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}.$$

Let

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix}.$$

Compute $E_1 A$, $E_2 A$, $E_3 A$.

3 (**this problem is optional!**) Let V be a vector space, $U \subset V$ a subspace and $\iota_U : U \to V$ the inclusion map:

$$\iota_U(u) = u \quad \text{for all } u \in U.$$

Prove that $\iota_U : U \to V$ is linear.

4 (**this problem is optional!**) Recall that given a function $f : X \to Y$ between two sets and a subset $Z \subset X$ the restriction $f|_Z : Z \to Y$ is defined by

$$f|_Z(z) := f(z)$$

for all $z \in Z$. (It may *seem* that there is no difference between $f|Z$ and f, but there is: the two functions have different domains of definition.) Now let V be a vector space, $U \subset V$ a subspace and $T : V \to W$ is a linear map. Prove that the restriction $T|_U$ is the inclusion map ι_U followed by T. That is, prove that

$$T|_U = T \circ \iota_U.$$

5 The point of this exercise is to explore direct sums of subspaces.

Definition Let V be a vector space, $W_1, W_2 \subset V$ two subspaces such that

1. $V = W_1 + W_2$ and
2. $W_1 \cap W_2 = \{0\}$.

We say that V is a direct sum of W_1 and W_2 and write $V = W_1 \oplus W_2$.

(a) Prove that $V = W_1 \oplus W_2$ if and only for every $v \in V$ there are unique $w_1 \in W_1$, $w_2 \in W_2$ such that $v = w_1 + w_2$.

(b) Suppose $V = W_1 \oplus W_2$. Prove that given any two linear maps $T_1 : W_1 \to U$, $T_2 : W_2 \to U$ there is a unique linear map $T : V \to U$ so that $T|_{W_1} = T_1$ and $T|_{W_2} = T_2$. (See problem **4** for reminders about restrictions.)

6 Let V be a finite dimensional vector space, $S, T : V \to V$ two linear maps such that $T \circ S = id_V$. Prove that $S \circ T = id_V$ as well.
Hint: Lemma 7.2 from lecture 7 may be useful.

7 Recall from Homework 3 problem 7 that for any set X and any vector space W the space of maps $\mathrm{Map}(X, W)$ from X to W is a vector space. Now take $X = \mathbb{N}$ (natural numbers) and $W = \mathbb{R}$, the vector space of real numbers. Then $V = \mathrm{Map}(\mathbb{N}, \mathbb{R})$ is a vector space. It is the space of sequences of real numbers. Define linear maps $S, T : V \to V$ to be two shifts:

$$S(a_1, a_2, \ldots, a_n, \ldots) := (0, a_1, a_2, \ldots, a_n, \ldots)$$

$$T(a_1, a_2, a_3, \ldots, a_n, \ldots) := (a_2, a_3, \ldots, a_n, \ldots).$$

Prove that $T \circ S = id_V$ and yet neither S nor T are invertible. Thus the assumption in problem 4 that V is finite dimensional is essential.

8 Consider the linear map $P : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$P \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Prove that

- $P \circ P = P$ and that
- $\mathbb{R}^n$ is the direct sum of $N(P)$ and $R(P)$ (direct sums are defined in problem 5).