Homework #3, Math 416, **Prof. Eugene Lerman**
Due Wednesday, February 6, 2019 in class

1 Let V be a vector space and S a subset of V. Prove that the span of S equals the intersection of all the subspaces of V containing S.

2 Find a necessary and sufficient condition on the real numbers x and y so that the vectors $(1, x)^T$ and $(1, y)^T$ in $\mathbb{R}^2$ are linearly independent.

3 Consider the two vectors $(0, 0, 1)^T$ and $(1, 1, 0)^T$ in $\mathbb{R}^3$. Find two different bases of $\mathbb{R}^3$ that contain these two vectors. Justify that they are a basis.

4 Prove that x, y, z are vectors in some vector space V with $x+y+z = \vec{0}$, then $\text{span}\{x, y\} = \text{span}\{y, z\}$.

5 Recall that $\mathcal{P}_n$ denotes the space of all polynomials of degree $\leq n$ and that the derivative $\frac{d}{dx} : \mathcal{P}_n \to \mathcal{P}_n$ is a linear map. Find the null space and range of $\frac{d}{dx}$. Show that the intersection of the null space of $\frac{d}{dx}$ with its range is not zero.

6 Suppose $T : V \to W$ is a linear map which is 1-1. Show that if $\{v_1, \dots, v_n\}$ is a basis of V, then the set $S = \{T(v_1), \dots, T(v_n)\}$ is linearly independent. Does the set S necessarily span W?

7a Let X be a set and W a vector space. Then any two functions from X to W can be added ($(f + g)(x) := f(x) + g(x)$ for all $x \in X$) and any function from X to W can be multiplied by scalars ($(\lambda f)(x) := \lambda(f(x))$ for all $x \in X$, $\lambda \in \mathbb{R}$). Prove that the set

$$\text{Map}(X, W)$$

of all maps (functions) from X to W is a vector space under the operations described above.

7b Now assume that X is also a vector space. Then we can consider the subset

$$\text{Hom}(X, W) := \{T : X \to W \mid T \text{ is linear }\}$$

of $\text{Map}(X, W)$ consisting of linear maps. Prove that $\text{Hom}(X, W)$ is a subspace of $\text{Map}(X, W)$. So in particular $\text{Hom}(X, W)$ is a vector space.

8 Prove that the linear maps $\ell_1, \ell_2, \ell_3 : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$\ell_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1, \quad \ell_2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2, \quad \ell_3 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3$$

form a basis of the space $\text{Hom}(\mathbb{R}^3, \mathbb{R})$ of linear maps from $\mathbb{R}^3$ to $\mathbb{R}$