Homework # 14, Math 416, **Prof. Eugene Lerman**
Due Wednesday, May 1, 2019 in class

Read pp 259 – 273 of Treil's book. In particular section 4.2 should be useful for computing Jordan Normal Form. Problem 7 will not be graded.

1 Let $\mathcal{A}$ be an associative algebra over $\mathbb{C}$. For example $\mathcal{A}$ is the algebra $M_{n,n}(\mathbb{C})$ of $n \times n$ matrices. Define the bracket

$$[\,\cdot\,,\,\cdot\,] : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

by

$$[a_1, a_2] := a_1 a_2 - a_2 a_1$$

for all $a_1, a_2 \in \mathcal{A}$. Prove

1. $[\,\cdot\,,\,\cdot\,]$ is bilinear;
2. $[\,\cdot\,,\,\cdot\,]$ is anti-symmetric: $[a_2, a_1] = -[a_1, a_2]$ for all $a_1, a_2 \in \mathcal{A}$;
3. the Jacobi identity holds:
$$[a_1, [a_2, a_3]] = [[a_1, a_2], a_3] + [a_2, [a_1, a_3]]$$
for all $a_1, a_2, a_3 \in \mathcal{A}$.

2 Consider the matrix $A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{pmatrix}$.

(a) Compute the characteristic polynomial of A.

(b) Compute the eigenvalues of A and find the corresponding eigenspaces E_λ.
Recall that $E_\lambda := \{v \in \mathbb{C}^4 \mid Av = \lambda v\}$.

(c) Is A diagonalizable? Explain.

3 Let $\mathcal{H}_n := \{A \in M_{n,n}(\mathbb{C}) \mid A^* = A\}$, the set of all $n \times n$ Hermitian matrices. Prove that $\mathcal{H}_n$ is naturally a vector space over $\mathbb{R}$. Is it a vector space over $\mathbb{C}$? Explain. Find a basis of $\mathcal{H}_n$ (as a vector space over $\mathbb{R}$) and compute the dimension of $\mathcal{H}_n$.
Hint: if you get stuck, consider first the case of $\mathcal{H}_2$.

4 Consider the matrix $A = \begin{pmatrix} -1 & 3 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & -1 \end{pmatrix}$.

(a) Compute the Jordan normal form of A.

(b) Find a matrix S so that SAS^{-1} is in Jordan normal form.

5 Find a unitary matrix U so that $U \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} U^{-1}$ is diagonal.

6 Consider the $\mathbb{C}$-linear map $S : M_{n,n}(\mathbb{C}) \to M_{n,n}(\mathbb{C})$ defined by taking the transpose:

$$S(A) = A^T.$$

Prove that S is diagonalizable.
Hint: $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$ for all square matrices A.

7 Find the Jordan normal form and the corresponding basis for the linear map

$$T : M_{2,2}(\mathbb{C}) \to M_{2,2}(\mathbb{C})$$

which is defined by

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 3a & 2b + c \\ 2c + b & 3d \end{pmatrix}.$$

Hint: the matrices $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ form a basis of $M_{2,2}(\mathbb{C})$. A choice of this basis identifies $M_{2,2}(\mathbb{C})$ with $\mathbb{C}^4$.