Homework # 13, Math 416, **Prof. Eugene Lerman**
typo in problem 1 fixed on 4/21, **6c** restated 4/22 **Due Wednesday, April 24, 2019 in class**

Read sections 6.1, 6.2 of Linear algebra done wrong. Solve the following problems.

1 Let A be an $m \times n$ and B a $k \times m$ complex matrices. Prove that

$$(BA)^* = A^* B^*.$$

Here as before * means conjugate transpose.

2 Suppose $T : V \to W$ and $S : W \to U$ are two complex linear maps between complex finite dimensional inner product spaces. Prove that

$$(S \circ T)^* = T^* \circ S^*.$$

Here * means the adjoint.

3 Let V be a finite dimensional complex vector space with an inner product and $\mathcal{B} = \{b_1, \dots, b_n\}$ an orthonormal basis of V.
(a) Prove that the unique linear map $\varphi : \mathbb{C}^n \to V$ that takes the standard basis $\{e_1, \dots, e_n\}$ of $\mathbb{C}^n$ to $\mathcal{B}$ is unitary.
(b) Let $\mathcal{B}'$ be another orthonormal basis of V. Prove that the change of coordinates matrix $[\mathrm{id}_V]_{\mathcal{B},\mathcal{B}'}$ is unitary.

4 A Hermitian matrix A is positive definite if

$$(v, Av) > 0$$

for all non-zero vectors v. Prove that a matrix A is positive definite if and only if all the eigenvalues of A are (real and) positive.

5 Prove that if a Hermitian matrix A is positive definite then there is a matrix S so that $A = S^* S$.
Hints:

(i) Since A is Hermitian, it is unitarily diagonalizable: there is a unitary matrix U so that $A = UDU^*$ where D is diagonal.

(ii) Argue that since A is positive definite, all the diagonal entries of D are postive real numbers. Explain why this implies that there is a diagonal matrix D' with real entries so that $D = D'D'$.

(iii) $A = UD'D'U^*$. What should you take to be the matrix S?

6 A $n \times n$ matrix A is skew-Hermitian if and only if $A^* = -A$.
(a) Prove that A is skew-Hermitian $\Leftrightarrow$ $\sqrt{-1}A$ is Hermitian
(b) Prove that if a matrix A is skew-Hermitian then all of its eigenvalues are purely imaginary (you may consider 0 as purely imaginary for the purpose of this problem).
(c) Prove that ~~one can rescale eigenvectors of a skew-Hermitian $n \times n$ matrix so that they form an orthonormal basis of $\mathbb{C}^n$~~ given a skew-Hermitian $n \times n$ matrix A there is an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of A.

7 A commutator of two $n \times n$ matrices A and B is the matrix $[A, B]$ defined by

$$[A, B] := AB - BA.$$

($[A, B]$ is also called the Lie bracket of A and B).
(a) Prove that

$$[B, A] = -[A, B]$$

for any two matrices A and B.
(b) Prove that if A and B are two skew-Hermitian matrices than their bracket $[A, B]$ is also skew-Hermitian.
(c) Prove that if A and B are Hermitian, then $[A, B]$ is **not** Hermitian.

8 Let A be an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding orthonormal eigenvectors $v_1, \ldots, v_n \in \mathbb{C}^n$.

(a) Prove that

$$I = \sum_{j=1}^{n} v_j v_j^*$$

where I is the identity matrix, v_j's are column vectors and v_j^*'s are the corresponding row vectors.

(b) Prove that

$$A = \sum_{j=1}^{n} \lambda_j v_j v_j^*.$$

(c) (Optional !) What do parts (a) and (b) say in the Dirac notation?