**Homework # 12, Math 416,** **Prof. Eugene Lerman**
**Due Wednesday, April 17, 2019 in class**

Keep reading chapter 5 of *Linear algebra done wrong*. Solve the following problems.

**1** Let $V$ be a complex vector space and $V_{\mathbb{R}}$ the corresponding real vector space (restrict scalar multiplication on $V$ to $\mathbb{R} \subset \mathbb{C}$ (see section 5.8 in Teil's book). Prove that if $\{v_1, \ldots, v_n\}$ is a basis of $V$ then $\{v_1, \sqrt{-1}v_1, \ldots, v_n, \sqrt{-1}v_n\}$ is a basis of $V_{\mathbb{R}}$. In particular $\dim V_{\mathbb{R}} = 2 \dim V$.

**2** Suppose $T : V \to W$ is an anti-linear **bijection** between two complex vector spaces. That is, suppose that

$$T(\lambda_1 v_1 + \lambda_2 v_2) = \bar{\lambda}_1 T(v_1) + \bar{\lambda}_2 T(v_2)$$

for all $v_1, v_2 \in V$, $\lambda_1, \lambda_2 \in \mathbb{C}$. Prove that if $\{v_1, \ldots, v_k\} \subset V$ is linearly independent then so is its image $\{T(v_1), \ldots, T(v_k)\}$ in $W$.

**3** The point of the problem is to give you an example of a nondegenerate "inner product" that is not positive definite. Such inner products show up in relativity. Consider the map

$$b : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}, \qquad b((x, y)^T, (x', y')^T) = xx' - yy'.$$

**Assume** that $b$ is bilinear (it is, but I am not asking you to prove it).
**(a)** Show that the map $b^{\#} : \mathbb{R}^2 \to (\mathbb{R}^2)^*$ defined by

$$\left(b^{\#}((x, y)^T)\right)((u, v)^T) = b((x, y)^T, (u, v)^T)$$

for all $(x, y)^T, (u, v)^T \in \mathbb{R}^2$ is an isomorphism of vector spaces. Hint: show that $b^{\#}(e_1) = e_1^*$ and $b^{\#}(e_2) = -e_2^*$ where $\{e_1, e_2\}$ is the standard basis and $\{e_1^*, e_2^*\}$ is the dual basis.
**(b)** Prove that there is a nonzero vector $v \in \mathbb{R}^2$ so that $b(v, v) = 0$. Show also that there are vectors $v', v'' \in \mathbb{R}^2$ so that $b(v', v') < 0$ and $b(v'', b'') > 0$.

**4** Let $V$ be the vector space of continuous real-valued functions on the interval $[-1, 1]$:

$$V = \{f : [-1, 1] \to \mathbb{R} \mid f \text{ is continuous } \}.$$

Assume that $(f, g) = \int_{-1}^{1} f(t)g(t)\, dt$ is a positive definite inner product.
**(a)** Find an orthonormal basis of the subspace $E = \mathrm{Span}(\{1, t, t^2\})$.
**(b)** Find the orthogonal projection of $f(t) = |t|$ onto the subspace $E$.

**5** Let $V$ be a finite dimensional Hermitian vector space, $E \subset V$ a subspace (with $0 < \dim E < \dim V$) and $P_E : V \to V$ the corresponding orthogonal projection.
**(a)** Find the eigenvalues and eigenvectors of $P_E$.
**(b)** Prove that trace of $P_E$ is $\dim E$.

**6** Let $P : V \to V$ be an orthogonal projection (i.e., $P^2 = P$ and $R(P) \perp N(P)$. Prove that for any two vectors $v, w \in V$

$$(P(v), w) = (v, P(w)).$$

In other words prove that $P = P^*$.

**7** An $n \times n$ complex matrix $A$ is called unitary iff $A^*A = I$.
**(a)** Prove that any unitary matrix $A$ is invertible and that $A^{-1} = A^*$.
**(b)** Prove that $A$ is unitary if and only if for any $v, w \in \mathbb{C}^n$

$$(Av, Aw) = (v, w).$$

Here $(v, w) = \sum v_i \bar{w}_i$ is the standard inner product on $\mathbb{C}^n$.
**(c)** Prove that a matrix $A$ is unitary if and only if its columns are orthonormal: for any two columns $a_i, a_j$ of $A$, $(a_i, a_j) = \delta_{ij}$.
**(d)** Prove that unitary $n \times n$ matrices form a group under the matrix multiplication (groups are mentioned in lecture 18). This group is usually denoted by $U(n)$ and is called the unitary group (in dimension $n$). In particular check that

- $I \in U(n)$
- For any $A, B \in U(n)$ the product $AB$ is in $U(n)$.
- For any $A \in U(n)$ the inverse $A^{-1}$ is also in $U(n)$.

**8** Prove that for any $n \times n$ matrix $A$

$$\det \bar{A} = \overline{\det A},$$

where, as usual, the matrix $\bar{A}$ is obtained from the matrix $A$ by taking the complex conjugates of the entries.