Homework # 11, Math 416, **Prof. Eugene Lerman**
sign errors in #6 fixed 4/7 **Due Wednesday, April 10, 2019 in class**

Start reading chapter 5 of *Linear algebra done wrong*. Solve the following problems.

1 Let $T : V \to V$ be a linear map. Prove that the set

$$W = \{v \in V \mid T^n(v) = 0 \text{ for some natural number } n\}$$

is a subspace of V.

2 Let V be a vector space with a Hermitian inner product $(\, , \,)$ and $\{v_1, \dots, v_n\}$ an orthonormal basis of V (so that $(v_i, v_j) = \delta_{ij}$).
(a) Prove that for any two vectors $x = \sum \alpha_i v_i$, $y = \sum \beta_j v_j$ in V,

$$(x, y) = \sum \alpha_k \bar{\beta}_k.$$

(b) Use part (a) to prove Parseval's identity:

$$(x, y) = \sum (x, v_k)\overline{(y, v_k)}.$$

3 Let V be a vector space with a Hermitian inner product $(\, , \,)$. Suppose $u, v \in V$ with $||u|| = 2$, $||v|| = 3$ and $(u, v) = 2 + i$. Compute

$$||u + v||^2, \qquad ||u - v||^2 \quad \text{and} \quad (u + iv, v + iu).$$

4 Consider the real vector space $V = M_{2,2}(\mathbb{R})$ of real 2×2 matrices. Is $(A, B) := \operatorname{tr}(AB)$ a (real) inner product on V? Prove your answer.

5 Let V be a complex finite dimensional vector space and let $\{b_1, \dots, b_n\}$ be a basis of V. Define a function $(\, , \,) : V \times V \to \mathbb{C}$ by

$$(\sum \alpha_i b_i, \sum \beta_j b_j) := \sum \alpha_k \bar{\beta}_k.$$

Prove that $(\, , \,)$ as defined above is a Hermitian inner product on the vector space V.

6 Let V be a Hermitian vector space with an inner product $(\, , \,)$ and the corresponding norm $|| \, ||$. Prove that for any vectors $x, y \in V$

$$(x, y) = \frac{1}{4}\left(||x + y||^2 - ||x - y||^2 - i||x - iy||^2 + i||x + iy||^2\right)$$

7 An $n \times n$ matrix A is called orthogonal iff $A^T A = I$.
(a) Prove that any orthogonal matrix A is invertible and that $A^{-1} = A^T$.
(b) Prove that A is orthogonal if and only if for any $v, w \in \mathbb{R}^n$

$$(Av, Aw) = (v, w).$$

Here $(v, w) = \sum v_i w_i$ is the standard inner product on $\mathbb{R}^n$.
(c) Prove that a matrix A is orthogonal if and only if its columns are orthonormal: for any two columns a_i, a_j of A, $(a_i, a_j) = \delta_{ij}$.
(d) Prove that the permutation matrices $\rho(\sigma)$, $\sigma \in S_n$ are orthogonal.
(e) Prove that orthogonal $n \times n$ matrices form a group under the matrix multiplication (groups are mentioned in lecture 18). This group is usually denoted by $O(n)$ and is called the orthogonal group in dimension n. In particular check that

- $I \in O(n)$
- For any $A \in O(n)$ the inverse A^{-1} is also in $O(n)$.