Homework # 10, Math 416, **Prof. Eugene Lerman**
Due Wednesday, April 3, 2019 in class

Read chapter 4 of *Linear algebra done wrong.* Solve the following problems.

1 Find characteristic polynomials, eigenvalues and eigenvectors of the following matrices:

$$\begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}, \qquad \begin{pmatrix} 4 & 1 & 2 & 3 \\ 0 & 3 & 4 & 3 \\ 0 & 0 & e & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2 Recall that a complex $n \times n$ matrix A is called nilpotent if $A^k = 0$ for some $k > 0$. Prove that if A is nilpotent then 0 is the only eigenvalue of A.

3 Let A be a square matrix with **real** enteries and let λ be a **complex** eigenvalue of A and $v = (v_1, \dots, v_n)^T$ the corresponding (complex) eigenvector. Prove that the complex conjugate $\bar{\lambda}$ is also an eigenvalue of A and that $\bar{v} := (\bar{v}_1, \dots, \bar{v}_n)^T$ is a corresponding eigenvector of A.

4 Let A be a complex $n \times n$ matrix.
(a) Does A^T have the same eigenvalues as A? Explain.
(b) Does in general A^T have the same eigenvectors as A? If not, give a counterexample.
(c) Suppose A is diagonalizable. Is A^T diagonalizable? Prove your answer.

5 Let $A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$. Find A^{2004} by diagonlizing A.

6
(a) Suppose A and B are two $n \times n$ matrices with $AB = BA$. Prove that for any $k > 1$

$$(AB)^k = A^k B^k.$$

(b) Prove that if A and B are two $n \times n$ matrices with $AB = BA$ then $(A + B)^N = \sum_{k=0}^{N} \binom{N}{k} A^k B^{N-k}$ for any $N > 0$.

(c) Find an expression for $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^k$.

Hint: $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

7 We defined the trace of a linear map $T : V \to V$ (where V is a real finite dimensional vector space) to be the trace of the matrix $[T]_{\mathcal{B}\mathcal{B}}$ with respect to some basis $\mathcal{B}$ of V:

$$\operatorname{tr}(T) := \operatorname{tr}([T]_{\mathcal{B}\mathcal{B}})$$

(a) Prove that $\operatorname{tr} : \operatorname{Hom}(V, V) \to \mathbb{R}$ is linear:

$$\operatorname{tr}(\lambda T + \mu S) = \lambda \operatorname{tr}(T) + \mu \operatorname{tr}(S)$$

for all $T, S \in \operatorname{Hom}(V, V)$ and all scalars λ, μ.
(b) Choose a vector $w \in V$ and a linear functional $\ell : V \to \mathbb{R}$.
Check that the map $T(v) := \ell(v)w$ is a linear map from V to V.
(c) Prove that $\operatorname{tr}(T) = \ell(w)$.

8 (a) Prove that two similar matrices have the same eigenvalues. Hint: if $A = SBS^{-1}$ for some invertible matrix S then $S(\lambda I - B)S^{-1} = S\lambda IS^{-1} - SBS^{-1} =$... and ...

(b) Let A be a non-zero nilpotent matrix. Prove that A cannot be diagonalized. That is, show that there is **no** invertible matrix S so that SAS^{-1} is diagonal.
Hint: problem 2 above should help you get started. Part (a) above should also be useful.