**1** Suppose $T : V \to W$ is an injective linear map between two vector spaces and $\{v_1, \ldots, v_n\} \subset V$ is linearly independent. Prove that $\{Tv_1, \ldots, Tv_n\} \subset W$ is linearly independent.

**2** Let $V$ be a complex vector space with a Hermitian inner product and suppose $\mathcal{B} = \{b_1, \ldots, b_n\}$ is an orthonormal basis. Prove that the map

$$T : \mathbb{C}^n \to V, \quad T \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} := \sum z_j b_j$$

is a surjective isometry, i.e., unitary.

**3** **(a)** Suppose $T : V \to W$ is a linear map, $\{v_1, \ldots, v_n\} \subset V$ and suppose that $\{Tv_1, \ldots, Tv_n\} \subset W$ is linearly independent. Prove that $\{v_1, \ldots, v_n\}$ is linearly independent.

**(b)** Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are two differentiable functions and that the determinant $D(x) := \det \begin{pmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{pmatrix}$ is a non-zero function (i.e., while the determinant may vanish for some values of $x$ it is not identically zero). Prove that $f$ and $g$ is linearly independent in the vector space $V$ of differentiable functions on $\mathbb{R}$.
Hint: fix $a \in \mathbb{R}$ such that $D(a) \neq 0$. Now consider the map $T : V \to \mathbb{R}^2$, $T(f) = (f(a), f'(a))^T$.

**4** Let $T : V \to W$ be a linear map between two finite dimensional vector spaces over $\mathbb{R}$, $\mathcal{A} = \{a_1, \ldots, a_n\}$ a basis of $V$, $\mathcal{B} = \{b_1, \ldots, b_m\}$ a basis of $W$, $\Phi_{\mathcal{A}} : \mathbb{R}^n \to V$ the isomorphism defined by

$$\Phi_{\mathcal{A}}((x_1, \ldots, x_n)^T) = \sum x_i a_i,$$

$\Psi_{\mathcal{B}} : W \to \mathbb{R}^m$ the isomorphism satisfying

$$\Psi_{\mathcal{B}}(\sum y_j b_j) = (y_1, \ldots, y_m)^T.$$

**(a)** Prove that $\Phi_{\mathcal{A}}$ sends the null space $N([T]_{\mathcal{B}\mathcal{A}})$ of the matrix $[T]_{\mathcal{B}\mathcal{A}}$ onto the null space $N(T)$.

**(b)** Prove that $\Psi_{\mathcal{B}}(R(T)) = R([T]_{\mathcal{B}\mathcal{A}})$.

**(c)** Now suppose that $\mathcal{A} = \{a_1, a_2, a_3\}$, $\mathcal{B} = \{b_1, b_2\}$ and that $[T]_{\mathcal{B}\mathcal{A}} = \begin{pmatrix} 3 & 9 & -6 \\ 1 & 3 & -2 \end{pmatrix}$. Find the basis of the null space $N(T)$ and of the range $R(T)$.

**5** Suppose $A$ is an $n \times n$ real matrix and there is a basis $\{b_1, \ldots, b_n\}$ of $\mathbb{R}^n$ so that $Ab_i = \lambda_i b_i$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.

**(a)** What is the characteristic polynomial of $A$?

**(b)** Suppose $v = \sum c_i b_i$ for some $c_1, \ldots, c_n \in \mathbb{R}$. Then $A^k v = \sum a_i b_i$ for some $a_i$'s. What are they?

**(c)** Suppose $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$. What is $A^k \begin{pmatrix} 1 \\ 2 \end{pmatrix}$?

**6** Suppose $V$ is a complex vector spaces of dimension $n$, $S, T : V \to V$ are two linear maps with $S \circ T = T \circ S$, $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $T$ and $\lambda_i \neq \lambda_j$ for $i \neq j$ (i.e., they are all distinct). Prove that $S$ is diagonalizable.

**7** Let $W_1, W_2$ be two subspaces of a finite dimensional vector space $V$. Suppose $\{c_1, \ldots, c_k\}$ is a basis of $W_1 \cap W_2$, $b_1, \ldots b_n \in W_2$ are vectors so that $\{c_1, \ldots, c_k, b_1, \ldots b_n\}$ is a basis of $W_2$, and $a_1, \ldots, a_\ell \in W_1$ are vector so that $\{c_1, \ldots, c_k, a_1, \ldots a_\ell\}$ is a basis of $W_1$. Prove that

$$\{a_1, \ldots, a_\ell, b_1, \ldots, b_n, c_1, \ldots, c_k\}$$

is a basis of $W_1 + W_2$.

**8** Let $V$ be a real finite dimensional vector space with an inner product and $E \subset V$ a subspace of $V$.

**(a)** Prove that

$$E^\perp := \{v \in V \mid (v, x) = 0 \text{ for all } x \in E\}$$

is a subspace of $V$.

**(b)** Prove that

$$E \cap E^\perp = \{0\}.$$

**(c)** Prove that

$$\dim E + \dim E^{\perp} = \dim V.$$

**9** Suppose $V$, $W$ are two finite dimensional complex vector spaces with Hermitian inner products. Suppose further that $\dim V \le \dim W$. Prove that there is an isometry $T : V \to W$.

**10** Give an example of two $2 \times 2$ matrices $A$ and $B$ that have the same eigenvalues and yet are not similar.

**11** Suppose $S, T : V \to V$ are two linear maps such that $S \circ T = 0 = T \circ S$ and $S + T = \mathrm{id}_V$. Prove that $V$ is a direct sum of the null spaces $N(S)$, $N(T)$:

$$V = N(S) \oplus N(T).$$

**12** Let $V$ be a complex finite dimensional vector space with a Hermitian inner product and $W \subset V$ a subspace and $P_W : V \to V$ the orthogonal projection onto $W$ (so that $R(P_W) = W$, $P_W^2 = P_W$ and $N(P_W) \perp R(P_W)$).
Prove that for any $v \in V$, $w \in W$

$$||v - P_W(v)|| \le ||v - w||.$$

**13** Suppose $V, W$ are two finite dimensional complex vector spaces with Hermitian inner products and $A : V \to W$ linear, $A^* : W \to V$ the adjoint of $A$, $y \in W$. Prove that for any $x \in V$

$$(Ax - y) \perp R(A) \quad \Leftrightarrow \quad A^*Ax = A^*y.$$

**14** Suppose $T : V \to V$ is a linear map, $\lambda \in \mathbb{C}$ and $\mathcal{B} = \{b_1, \ldots, b_n\}$ a basis of $V$ so that $Tb_1 = \lambda b_1$ and

$$Tb_k = \lambda b_k + b_{k-1} \qquad \text{for } k > 1.$$

What is the Jordan normal form of the map $T$?

**15** **(a)** State the Cauchy-Schwarz inequality.

**(b)** Use Cauchy-Schwarz to prove the triangle inequality.

**16a** Let $W$ be a subspace of a vector space $V$. Suppose $\{b_1, \ldots, b_k\}$ is a basis of $W$ and $\{b_1, \ldots, b_k, b_{k+1} \ldots b_n\}$ is a basis of $V$. Prove that $\{b_{k+1} + W, \ldots, b_n + W\}$ is a basis of $V/W$.

**16b** Let $T : V \to W$ be a linear map, $Z = N(T)$ the null space of $T$. Prove that $V/Z$ is isomorphic to the range $R(T)$ of $T$.

**17** Let $A = \begin{pmatrix} B & O \\ O & C \end{pmatrix}$ be a square block-diagonal matrix (with $B, C$ being square matrices of appropriate sizes). Prove that $\det A = \det B \det C$.

**18** **(a)** Let $V$ be a vector space. Suppose $\alpha : \overbrace{V \times \cdots \times V}^{k} \to \mathbb{R}$ is $k$-linear and alternating and $\{v_1, \ldots, v_k\} \subset V$ is a linearly dependent set. Prove that $\alpha(v_1, \ldots, v_k) = 0$.

**(b)** Prove that if $A$ in an $n \times n$ matrix which is not invertible then $\det A = 0$.

**19** Suppose $A, B$ are two $n \times n$ matrices. Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Hint: brute force.

**20** Suppose $T : V \to V$ is a linear map with the characteristic polynomial $\det(x\,\mathrm{id}_V - T) = (x-2)^3(x-3)^4$.

**(a)** What are the determinant and trace of $T$? Explain.

**(b)** Suppose $\dim N(T - 2\,\mathrm{id}_V) = 3 = \dim N(T - 3\,\mathrm{id}_V)$. Does this determine the Jordan normal form of $T$? If it does, what is it? If it doesn't, what are the possibilities?

**21** Prove that an upper triangular Hermitian matrix is diagonal.

**22** Suppose $V$ is a real vector space with a positive definite inner product, $v, w \in V$ two nonzero vectors with $(v, w) = 0$. Prove that $\{v, w\}$ is a linearly independent set.