Last time:

We defined a vector space V to be finite dimensional if (and only if) there is a finite set \( \{v_1, \ldots, v_m\} \subseteq V \), which forms a basis of V.

That is:

\[
V \text{ is finite dimensional } \iff V \text{ has a finite basis.}
\]

We proved:

Thm 1. If V is finite dimensional then any basis has the same number of elements as any other basis.

We therefore can define the dimension of V to be the number of elements in a basis of V.

We write \( \dim V = \# \text{ of elements in a basis of V} \).

Ex: \( 1, x, x^2, \ldots, x^n \) is a basis of \( \mathbb{R}^n \).

\( \Rightarrow \dim \mathbb{R}^n = n+1 \).

Compare this Thm with Prop 3.3 on p.49 of text.

(There the assumption that \( \dim V \) is finite is left out.)

Prop 2.8 (p.10): Let V be a vector space. Suppose \( S = \{v_1, \ldots, v_m\} \) spans V. Then there is a subset \( S' \) of \( S \) which forms a basis of V.

Proof: If \( S \) is linearly independent, we're done.

Otherwise one of the \( v_i \)'s is a linear combination of the rest. Say

\[
v_m = \sum_{i=1}^{m-1} \alpha_i v_i \quad \text{for some } \alpha_i \in \mathbb{F}.
\]

If \( x = a_1 v_1 + \ldots + a_m v_m \), then

\[
x = a_1 v_1 + \ldots + a_{m-1} v_{m-1} + a_m (\sum_{i=1}^{m-1} \alpha_i v_i)
\]
\[ (a_1 + a_m x_1) v_1 + \ldots + (a_{m-1} + a_m x_{m-1}) v_{m-1} \]

\[ \Rightarrow S' = \{ v_1, \ldots, v_{m-1} \} \text{ still spans } V \]

If \( S' \) is linearly independent, we're done — it's a basis.

Otherwise, proceed as before.

At every step we get a smaller set that spans \( V \). The process stops when we're left with a basis.

Since there are only \( m \) vectors in \( S \), the process cannot go on for more than \( m \) steps.

\[ \square \]

**Consequences:**

1. If \( v_1, \ldots, v_m \) spans \( V \), then \( m \geq \dim V \). (cf. 3.5 p. 49)
2. \( V \) in finite dimensional \( \Rightarrow \) \( V \) is spanned by a finite set.
   - (cf. prop 5.1, p. 54)
   - **Proof** (\( \Rightarrow \)): \( V \) finite dim \( \Rightarrow \) \( V \) has a finite basis.
     This basis in our finite set.
   - (\( \Leftarrow \)) above.
3. Prop 5.3 on p. 55: any generating system of a finite dimensional vector space has at least \( \dim V \) vectors in it. That's just another way of stating: ch1 prop 2.8.

Recall last time we also proved

**Lemma 2** If \( \{ z_1, \ldots, z_m \} \) is a set of linearly dependent vectors in a vector space \( V \), then there is \( k \) with \( 2 \leq k \leq m \) and \( z_k \) is a linear combination of \( z_1, \ldots, z_{k-1} \).

We'll use it to prove

**Prop 5.4** (p. 56) Any linearly independent set in a finite dimensional vector space can be completed to a basis.
That is, if \( \{v_1, \ldots, v_k \} \subseteq V \) is linearly independent, then \( \{v_k, \ldots, v_m \} \subseteq V \) so that \( \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_m \} \) is a basis of \( V \).

**Proof (different from text)**

Since \( V \) is finite dimensional it has a basis \( \{x_1, \ldots, x_n\} \).

We consider the set \( S \) of vectors

\[
V_1, \ldots, V_k, x_1, \ldots, x_n
\]

(\( x_1, \ldots, x_n \) in this order!)

and apply lemma several times in a row.

The set \( S \) is linearly dependent since \( v_i \)'s are linear combination of the basis vectors \( x_1, \ldots, x_n \).

By lemma, some vectors in \( S \) are linear combination of preceding ones. Let \( z \) be the first such vector.

Then \( z \) is different from any \( v_i \)'s (\( u \)'s are linearly independent!). Hence \( z = x_i \) for some \( i \), now consider the set \( S' \):

\[
S' = \{ v_1, \ldots, v_k, x_1, \ldots, x_{i-1}, x_i+1, \ldots, x_n \}
\]

This set still spans \( V \), since we can express \( x_i \) as a linear combination of \( v_1, \ldots, v_k, x_1, \ldots, x_{i-1} \) and \( x_i, x_i+1, \ldots, x_n \) spans \( V \).

Note: \( S' \) has \( n+m-1 \) vectors.

So unless \( m=1 \), it cannot be a basis—it's too big.

So it's linearly dependent.

Repeat the argument, and discard another \( x_i \).

We get a set \( S'' \) with \( n+m-2 \) vectors, which spans \( V \).

Unless \( n+m-2 = n \), we have too many vectors in \( S'' \) to be a basis. So \( S'' \) is linearly dependent.

Repeat. Eventually we get a set

\[
S^{(n-m)}
\]

that spans \( V \) and has exactly \( n = dm \) \( V \) vectors.
They have to be linearly independent, for otherwise we'd have n-1 vectors spanning V, which would contradict consequence (1).

**Corollary 3** (compare prop 5.2 on p.54)
Any linearly independent set in a finite dimensional vector space V has at most \( \dim V \) vectors in it.

Finally we expect the following to be true:

**Thm 4** If V is a finite dimensional vector space and U \( \subseteq \) V a subspace then U is finite dimensional and \( \dim U \leq \dim V \).

**Proof** Note that any basis \( \{v_1, \ldots, v_k\} \) of U is a linearly independent set in V hence \( k \leq \dim V \), by Corollary 3.
It remains to show that U has a basis.
If \( U = \{0\} \), it has an empty basis and \( \dim U = 0 \).
If \( U \neq \{0\} \), then it has at least 1 non-zero vector \( v_1 \).
If \( \text{span}\{v_1\} = U \), we're done.
If \( \text{span}\{v_1\} \neq U \), then \( \exists v_2 \in U \), \( v_2 \neq 0 \) and \( v_1, v_2 \) are linearly independent.
If \( \text{span}\{v_1, v_2\} = U \), we're done.
Otherwise \( \exists v_3 \neq 0, v_3 \in U \), \( v_3 \notin \text{span}\{v_1, v_2\} \).
The process stops when we get a basis of U.
It has to stop after \( \dim V \) steps.