1. Let $V, W$ be finite-dimensional vector spaces. Show that $V$ is isomorphic to $W$ if and only if $V$ and $W$ have the same dimension.

2. Let $T: V \rightarrow W$ be a linear map, $\{v_1, v_2, v_3\}$ a basis of $V$, and $\{w_1, w_2\}$ a basis of $W$ such that the matrix representing $T$ with respect to those bases is

$$[T]_{\{w_i\}}^{\{v_i\}} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \end{bmatrix}.$$  

Find a basis of $\ker T \subset V$ and a basis of $\im T \subset W$.

3. Compute $A^n$ where $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$.

4. Let $V, W$ be $n$-dimensional complex inner product spaces. Show that $V$ and $W$ are isometrically isomorphic.

5. Let $V$ be a finite-dimensional complex vector space and $\{v_1, \ldots, v_n\}$ a basis of $V$. For any vectors $v, w \in V$, express them in the given basis

$$v = \sum_i \alpha_i v_i, \quad w = \sum_i \beta_i v_i$$

and define

$$(v, w) := \sum_i \alpha_i \overline{\beta_i} \in \mathbb{C}.$$ 

Show that this formula $(v, w)$ defines a complex inner product on $V$.

6. Consider the plane $P$ in $\mathbb{R}^3$ given by the equation

$$2x_1 - 4x_2 + x_3 = 1.$$ 

Note that the plane does not go through the origin.

Find the distance from the point $\begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}$ to the plane.

7. Consider the real inner product space

$$C[-1, 1] := \{f: [-1, 1] \rightarrow \mathbb{R} \mid f \text{ is continuous} \}$$

with its usual inner product

$$(f, g) = \int_{-1}^{1} f(t)g(t) \, dt.$$ 

Consider the absolute value function $f(t) = |t|$. Find $\Proj_{\{1, t, t^2\}}(f)$.

8. Consider the quadratic form on $\mathbb{R}^2$ given by

$$Q(x) = 4x_1 x_2 + 3x_2^2.$$
Find the unique symmetric matrix $A$ satisfying $Q(x) = (Ax, x)$. Is $A$ (and hence $Q$) positive definite? Explain.

9 Prove that a complex $n \times n$ matrix $A$ is diagonalizable if and only if there exists a basis of $\mathbb{C}^n$ consisting of eigenvectors of $A$.

10 Let $N : V \rightarrow V$ be a nilpotent linear map on a finite dimensional complex vector space. Prove that the only eigenvalue of $N$ is 0. Does $N$ have to be the zero map? If no, give an example of a nonzero nilpotent linear map.

11 Suppose $V = U \oplus W$ (direct sum) and $T : V \rightarrow V$ is a linear map with $T(U) \subset U$, $T(W) \subset W$.
   a) Suppose further that the restrictions $T|_U$ and $T|_W$ are diagonalizable. Prove that $T$ is diagonalizable.
   b) What is the relation between $\det T$ and $\det(T|_U)$ and $\det(T|_W)$?

12 State the Cauchy-Schwarz inequality. Use Cauchy-Schwarz to prove the triangle inequality.

13 Let $V$ be a complex vector space of dimension 1. Prove that for any linear map $T : V \rightarrow V$ is a complex number $\lambda \in \mathbb{C}$ so that $T(v) = \lambda v$ for all $v \in V$.

   Give an example of two $2 \times 2$ matrices $A$ and $B$ which have the same eigenvalues but are not similar.

14 (deleted).  

15 State the Sylvester criterion. Is the matrix \[
\begin{pmatrix}
1 & 2 & 1 \\
2 & 9 & 1 \\
1 & 1 & 1
\end{pmatrix}
\] positive definite?

16 Suppose $A$ and $B$ are two Hermitian matrices with the same eigenvalues (counted with multiplicities). Are $A$ and $B$ similar? If yes, prove it. If not, give a counterexample.

17 Give an example of two $2 \times 2$ matrices $A$ and $B$ that have the same eigenvalues (with multiplicities) but are not similar.

18 Let $T : V \rightarrow W$ be a linear map and $U \subset V$ a subspace. Prove that if the restriction $T|_U$ of $T$ to $U$ is zero, then there is a well-defined linear map $\bar{T} : V/U \rightarrow W$ with $\bar{T}(v + U) = T(v)$ for all $v \in V$.

19 Suppose $S, T : V \rightarrow V$ are two linear maps on a finite dimensional vector space $V$ with $S \circ T = 0 = T \circ S$ and $S + T = I$. Prove that $V = \ker S \oplus \ker T$. 

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