1a. If \( v_1, v_2 \in Y + Z \) then \( \exists y_1, y_2 \in Y, z_1, z_2 \in Z \) so

\[ v_1 = y_1 + z_1, \quad v_2 = y_2 + z_2 \]

\[ \Rightarrow \forall a_1, a_2 \in \mathbb{R} \]

\[ a_1 v_1 + a_2 v_2 = a_1 (y_1 + z_1) + a_2 (y_2 + z_2) \]

\[ = (a_1 y_1 + a_2 y_2) + (a_1 z_1 + a_2 z_2) \]

Since \( Y \) is a subspace \((a_1 y_1 + a_2 y_2) \in Y \)

Since \( Z \) is a subspace \((a_1 z_1 + a_2 z_2) \in Z \)

\[ \Rightarrow a_1 v_1 + a_2 v_2 = \underbrace{(a_1 y_1 + a_2 y_2)}_{Y} + \underbrace{(a_1 z_1 + a_2 z_2)}_{Z} \in Y + Z \]

\[ \Rightarrow \underbrace{Y + Z}_{\text{a subspace of } V} \]

1b. Since \( V \) is finite dimensional, so are \( Y, Z, Y + Z \) and \( Y + Z \). Choose a basis \( \{ x_1, \ldots, x_k \} \) of \( Y + Z \).

Complete it to a basis \( \{ x_1, \ldots, x_k, y_{k+1}, \ldots, y_n \} \) of \( Y \)

\[ \{ x_1, \ldots, x_k, y_{k+1}, \ldots, y_n, z_{k+1}, \ldots, z_m \} \] of \( Z \).

\[ \text{Claim:} \quad \{ x_1, \ldots, x_k, y_{k+1}, \ldots, y_n, z_{k+1}, \ldots, z_m \} \text{ is a basis of } \]

\[ Y + Z \]

Indeed, if \( v \in Y + Z \) then \( v = y + z \) for some \( y \in Y, z \in Z \).

Since \( \{ x_1, \ldots, x_k, y_{k+1}, \ldots, y_n \} \) is a basis of \( Y \)

\[ v = a_1 x_1 + \ldots + a_k x_k + a_{k+1} y_{k+1} + \ldots + a_n y_n \]

for some scalars \( a_1, \ldots, a_n \).

Similarly \( z = \beta_1 x_1 + \ldots + \beta_k x_k + \beta_{k+1} y_{k+1} + \ldots + \beta_m y_m \)

for some scalars \( \beta_1, \ldots, \beta_m \)

\[ \Rightarrow v = y + z = (a_1 + \beta_1) x_1 + \ldots + (a_k + \beta_k) x_k + a_{k+1} y_{k+1} + \ldots + a_n y_n + \beta_{k+1} z_{k+1} + \ldots + \beta_m z_m \]

\[ \Rightarrow \{ x_1, \ldots, x_k, y_{k+1}, \ldots, y_n, z_{k+1}, \ldots, z_m \} \text{ spans } Y + Z \]

Linear independence.

Suppose 3 scalars \( a_1, \ldots, a_n, \beta_{k+1}, \ldots, \beta_m \) so that

\[ 0 = a_1 x_1 + \ldots + a_k x_k + a_{k+1} y_{k+1} + \ldots + a_n y_n + \beta_{k+1} z_{k+1} + \ldots + \beta_m z_m \]
Then \[-(\beta_{k+1} + \cdots + \beta_m z_m) = \alpha_1 x_1 + \cdots + \alpha_k x_k + \alpha_{k+1} y_{k+1} + \cdots + \alpha_m y_m \in \mathbb{Y} \cap \mathbb{Z} = \text{span} \{y_1, x_k\}.\]

Hence \(3 \beta_i = \beta_k\), so that
\[\alpha_1 x_1 + \cdots + \alpha_k x_k + \alpha_{k+1} y_{k+1} + \cdots + \alpha_m y_m = Y x_1 + \cdots + \beta_k x_k\]

But \(x_1, x_k, y_{k+1}, \ldots, y_m\) is linearly independent.
\[\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0, \quad \alpha_{k+1} = 0, \ldots, \alpha_m = 0.\]

\[\Rightarrow (-\beta_{k+1}) z_{k+1} + \cdots + (-\beta_m) z_m = \alpha_1 x_1 + \cdots + \alpha_k x_k\]

But \(x_1, x_k, \beta_{k+1}, \ldots, \beta_m\) is linearly independent.
\[\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0, \quad \beta_{k+1} = \beta_{k+2} = \cdots = \beta_m = 0.\]

So, \(x_1, x_k, y_{k+1}, \ldots, y_m, \beta_{k+1}, \ldots, \beta_m\) is a basis of \(Y + \mathbb{Z}\).
\[\Rightarrow \dim (Y + \mathbb{Z}) = n + (m-k) = \dim Y + \dim \mathbb{Z} - \dim (Y \cap \mathbb{Z}).\]

3a. For any two \(W\)-valued functions \(f, g : X \to W\)
we define addition by \ addition \(f + g\) \(X\)
\[(f + g)(x) = f(x) + g(x) \forall x \in X\]

We define mult by scalars by \(c \cdot f\) \(X\)
\[(c \cdot f)(x) = c \cdot f(x) \forall x \in X\text{ scalar \ mult. in } W.\]

One checks that Map(X, W) is a vector space with + and scalar multiplication as defined above.

3b. We need to check that the set of linear maps \(\text{Hom}(X, W)\) is closed under + and scalar multiplication in Map(X, W).
\[ T, S : X \to W \text{ are linear}, \]
\[ \alpha, \beta \text{ scalars, then } \forall x_1, x_2 \in X, \]
\[ (\alpha T + \beta S)(x_1 + x_2) = \alpha T(x_1 + x_2) + \beta S(x_1 + x_2) \]
\[ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{(by def of } + \text{ and scalar mul. on Map}(X, W)) \]
\[ = \alpha (T(x_1) + T(x_2)) + \beta (S(x_1) + S(x_2)) \]
\[ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{(since } T \text{ and } S \text{ are linear}) \]
\[ = (\alpha T(x_1) + \beta S(x_1)) + (\alpha T(x_2) + \beta S(x_2)) \]
\[ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{(since } W \text{ is a vector space}) \]
\[ = (\alpha T + \beta S)(x_1) + (\alpha T + \beta S)(x_2) \]

Similarly, \( \forall \) scalar \( \alpha \) and any \( x \in X \)
\[ (\alpha T + \beta S)(\alpha x) = \alpha T(\alpha x) + \beta S(\alpha x) \]
\[ = \alpha \alpha T(x) + \beta \alpha S(x) \]
\[ = \alpha (\alpha T(x) + \beta S(x)) \]
\[ = \alpha . (\alpha T + \beta S)(x) \]
\[ \Rightarrow \alpha T + \beta S \text{ is linear.} \]
\[ \Rightarrow \text{Hom}(X, W) \text{ is a subspace of Map}(X, W). \]