We fill in the details left over from March 31 notes.

1. The tautological 1-form $\alpha \in \Omega^1(T^*M)$.

Recall that in coordinates $(q_1, q_n): M \subset \mathbb{R}^n$ the corresponding coordinates $(q_1, q_n, p_1, p_n): T^*U \to \mathbb{R}^n \times \mathbb{R}^n$ the tautological 1-form $\alpha = dq^m$ is given by

$$\alpha = \sum p_i dq_i.$$

On the other hand, $\alpha$ has the following coordinate-free description. Suppose $x \in M$, $\eta \in T^*_x M$, $v \in T_x(T^*M)$. Then $d\pi_x : T^*_x M \to T^*_x M$ where $\pi_x : T^*M \to M$

in the canonical projection.

We set

$$\alpha(x, \eta)(v) = \langle \eta, (d\pi_x)_x (v) \rangle,$$

where $\langle \cdot, \cdot \rangle : T^*_x M \times T_x M \to \mathbb{R}$ in the canonical pairing.

**Lemma** The two definitions of $\alpha$ agree.

**Proof** Let $(q_1, q_n) : U \to \mathbb{R}^n$, $(q_1, q_n, p_1, p_n) : T^*U \to \mathbb{R}^n \times \mathbb{R}^n$ be coordinate charts as above, let $x \in U$, $\eta \in T^*_x U = T^*_x M$, $v \in T_x(T^*M)$.

Then

1. $\eta = \sum p_i(\eta)(dq_i)_x$, by definition of the induced coordinates $(q_1, q_n, p_1, p_n)$.

2. $v = \sum_{i=1}^n \left( v_i \frac{\partial}{\partial q_i} |_{(x, \eta)} + v_i n \frac{\partial}{\partial p_i} |_{(x, \eta)} \right)$

for some $v_1, \ldots, v_n + i\mathbb{R}$. 

April 15, 2013
Claim \[ d \pi(x, \eta) \left( \frac{\partial}{\partial p_i} \bigg|_{(x, \eta)} \right) = \frac{\partial}{\partial p_i} \bigg|_{x} \eta \bigg] \]
\[ d \pi(x, \eta) \left( \frac{\partial}{\partial q_i} \bigg|_{(x, \eta)} \right) = 0 \quad \forall \eta. \]

Exercise: prove the claim.

We now compute
\[ d \pi(x, \eta) (\nu) = \langle \nu, (d \pi)(x, \eta)(\nu) \rangle = \langle \sum \pi_i(\eta) (d f_i) \bigg|_{x}, \sum \nu_j \frac{\partial}{\partial q_j} \bigg|_{x} + \nu_{x+} \frac{\partial}{\partial p_{x+}} \bigg|_{(x, \eta)} \rangle \]
\[ = \langle \sum \pi_i (d f_i)(x, \eta), \sum \nu_j \frac{\partial}{\partial q_j} \bigg|_{(x, \eta)} + \nu_{x+} \frac{\partial}{\partial p_{x+}} \bigg|_{(x, \eta)} \rangle \]
\[ = (\sum \pi_i d f_i)(x, \eta) (\nu). \quad \square \]

Recall: if \( \varphi : M \to M \) is any diffeomorphism, we can lift it to a diffeomorphism \( \tilde{\varphi} : T^*M \to T^*M \).

For \( x \in M, \gamma \in T^*_xM \)
\[ (\ast) \quad \tilde{\varphi}(x, \eta) = \varphi(x), (d \varphi^{-1})^T_{\varphi(x)} \eta \]

Note: By definition (\ast) of \( \tilde{\varphi} \)
\[ \pi \circ \tilde{\varphi} = \varphi \circ \pi \]
where \( \pi : T^*M \to M \) is the canonical projection.

Consequence: Let \( G \times M \to M \) be an action of a Lie group \( G \) on a manifold \( M \), \( G \times T^*M \to T^*M \) the corresponding lifted action (see p.10.3), \( x \circ g \in \text{Lie}(G) \) a vector and \( X_M, X_{T^*M} \) the corresponding induced vector fields on \( M \) and \( T^*M \), respectively.

Proposition \( d \pi \circ X_{T^*M} = X_M \circ \pi^* \), where, as before,
\[ \pi : T^{*}M \to M \text{ is the canonical projection.} \]

**Proof** By definition of a lifted action, for any \( g \in G, x \in M, \eta \in T^{*}M, \)
\[ \pi (g \cdot (x, \eta)) = g \cdot x \quad (= g \cdot \pi (x, \eta)) \]
\[ \Rightarrow \quad d\pi (X_{T^{*}M} (x, \eta)) = \frac{d}{dt} \bigg|_{t=0} \pi ((\exp tX) \cdot (x, \eta)) \]
\[ = \frac{d}{dt} \bigg|_{t=0} \exp tX \cdot \pi (x, \eta) \quad \text{by} \]
\[ = X_M (\pi (x, \eta)) \quad (= X_M (x)) \quad \Box \]

**Proposition 2** For any diffeomorphism \( \varphi : M \to M \)
\[ (\varphi)^{*} \alpha_{T^{*}M} = \alpha_{T^{*}M}, \]
where \( \tilde{\varphi} : T^{*}M \to T^{*}M \) is the lift of \( \varphi, \) and
\[ \alpha_{T^{*}M} = \alpha \in \Omega^1 (T^{*}M) \text{ is the tautological 1-form.} \]

**Proof** We use the fact that \[ \Theta \circ \pi = \varphi \circ \pi. \]
\[ \forall x \in M, \eta \in T_{x}^{*}M, \upsilon \in T_{(x, \eta)} (T^{*}M) \]
\[ (\varphi \circ \pi)^{*} (\upsilon) = \alpha_{\varphi (x, \eta)} \circ \left((d \tilde{\varphi})_{(x, \eta)} \circ \upsilon\right) \quad \text{by def of } \alpha^* \]
\[ = \langle (d \varphi^{-1})_{(\varphi (x))} \circ d \pi \tilde{\varphi}_{(x, \eta)} \circ d \tilde{\varphi}_{(x, \eta)} \circ \upsilon, \upsilon \rangle \quad \text{by def of } d \pi \tilde{\varphi}_{(x, \eta)} \]
\[ = \langle \eta, (d \varphi^{-1})_{(\varphi (x))} \circ d \varphi \circ d \pi \tilde{\varphi}_{(x, \eta)} \circ \upsilon, \upsilon \rangle \quad \text{by } \Theta \]
\[ = \langle \eta, d \pi (x, \eta) \upsilon, \upsilon \rangle \quad \text{by chain rule (and the fact that} \]
\[ (\varphi \circ \pi) (x) = x \]
\[ = \alpha (x, \eta) (\upsilon) \quad \text{by definition of } \alpha. \quad \Box \]

(Proposition 2 is fact #3 on p. 106.) We use it together with Cartan's formula to prove:
\[ \forall \gamma \in \mathfrak{g}, \]
\[ M^{\gamma} : T^{*}M \to M, \quad M^{\gamma} (x, \eta) = \alpha (x, \eta) \left((\exp t \mathfrak{g}) (x, \eta)\right) \]
is a conserved quantity for any $G$-invariant Hamiltonian $H: T^*M \to \mathbb{R}$.

Proposition 1 implies:

**Proposition 3** let $G \times M \to M$, $G \times T^*M \to T^*M$ be the actions as in Proposition 1, and let

$$M^y(x, \eta) = \alpha(y, \eta) (Y_{T^*M}(x, \eta))$$

Then

$$M^x(x, \eta) = \langle \eta, Y_M(x) \rangle.$$

**Proof**

$$\langle \alpha(x, \eta) (Y_{T^*M}(x, \eta)), \langle \eta, Y_{T^*M}(x, \eta) \rangle \rangle$$

$$= \langle \eta, Y_M(x) \rangle$$

by Prop 1.

This concludes our proof of Noether's theorem (except for a proof of Cartan's formula).