Tangent spaces

If \( \Sigma \subseteq \mathbb{R}^3 \) is a surface, and \( p \in \Sigma \) a point, we know how to define the tangent space \( T_p \Sigma \) to \( \Sigma \) at \( p \); it's the space of "infinitesimal displacements"

\[
T_p \Sigma = \{ \dot{\gamma}(0) \mid \gamma : I \to \Sigma \text{ a curve, } \gamma(0) = p \}
\]

There are two issues with the definition:

(i) It's not clear that \( T_p \Sigma \) is a vector space
(ii) It's not obvious how to generalize this definition for an abstract manifold \( M \).

In math 423 we got around (i) by picking a parametrization \( \Phi : U \to V \cap \Sigma \) (\( U \subseteq \mathbb{R}^2 \), \( V \subseteq \mathbb{R}^3 \) open, \( p \in V \cap \Sigma \)) and showing that

\[
T_p \Sigma = \{ D\Phi(q)v \mid v \in \mathbb{R}^2 \}
\]

where \( q = \Phi^{-1}(p) \). That is, for any curve \( \gamma : I \to \Sigma \) with \( \gamma(0) = p \), there is a vector \( v \in \mathbb{R}^2 \) so that

\[
\gamma'(0) = D\Phi(q)v
\]

This is not hard to generalize if our manifold \( M \) is sitting inside some \( \mathbb{R}^n \). More formally, if \( M \) is embedded in \( \mathbb{R}^n \). This means: \( M \subseteq \mathbb{R}^n \) and

\( \forall p \in M \exists \text{ an open set } V \subseteq \mathbb{R}^n \text{ containing } p \) and a diffeomorphism \( \Phi : V \to W \subseteq \mathbb{R}^n \) (\( W \) open) so that

\[
\Phi(V \cap M) = W \cap (\mathbb{R}^n \times 0^n) = \{ (x_1, \ldots, x_n) \mid x_n = 0 \}
\]

(i.e. here \( M \) has dimension \( e \) and \( \Phi^{-1} \mid Wn(\mathbb{R}^e \times 0^n) \))

This will give a local parametrization of \( M \).

We then set \( q = \Phi(p) \) and define

\[
T_p M = \{ D\Phi^{-1}(q)v \mid v \in \mathbb{R}^e \times 0^n \}
\]
Lemma: With the notation above,
\[ T_p M = \{ \dot{\gamma}(0) \mid \gamma : I \rightarrow M, \gamma(0) = p \} \]

Proof. Given \( \gamma : I \rightarrow M \), let \( \dot{\gamma} = \dot{\gamma}(0) \) and \( \nu = \dot{\gamma}(0) \).

Then \( D(\dot{\gamma}\'')(t)u = \frac{d}{dt} \bigg|_{t=0} (\dot{\gamma}'(t)) = \frac{d}{dt} \bigg|_{t=0} (\dot{\gamma}'(0)) = \dot{\gamma}(0) \).

Why is this not completely satisfactory?

(1) In this setup, \( T_p M \) is a subspace of \( \mathbb{R}^n \) and it seems to depend on how \( M \) sits inside \( \mathbb{R}^n \).

(2) What do we do when \( M \) is given abstractly, and not as a subspace of some \( \mathbb{R}^n \)?

What do we do if \( M \) is given as a subset of two different \( \mathbb{R}^n \)'s?

There are two equivalent solutions to the problem and one usually uses both:

(i) tangent vectors are equivalent classes of curves (they are "infinitesimal displacements")

(ii) tangent vectors are abstract "directional derivatives" called derivations.

Let \( M \) be a manifold, \( p \in M \).

Definition. Two curves \( \gamma, \tau : I \rightarrow M \), \( \gamma(0) = \tau(0) = p \) are tangent to each other at \( p \) if for any \( f \in C^\infty(M) \)
\[ \frac{d}{dt} \bigg|_{t=0} f(\gamma(t)) = \frac{d}{dt} \bigg|_{t=0} f(\tau(t)). \]

We write: \( \gamma \sim \tau \)

We'd like to think of \( \gamma \) and \( \tau \) as representing the same infinitesimal displacement at \( p \).

Formally, \( \sim_p \) is an equivalence relation on curves.
in $M$ passing through $p$ and we define a tangent vector $v$ to $M$ at $p$ to be the equivalence class of such curves:

$$v = [\gamma].$$

Problem It's not at all clear that these "displacements" form a vector space.

However, for any smooth function on $M$ it makes sense to differentiate $f$ in the direction of $v = [\gamma]$:

$$(\mathbf{d}) \quad v(f) := \frac{d}{dt} \bigg|_{t=0} f(\gamma(t))$$

By definition of $v$, it doesn't matter which curve in $v$ we use to compute the R.H.S. of $(\mathbf{d})$.

Note that $(\mathbf{d})$ defines a **linear map** $v : C^0(M) \to \mathbb{R}$.

Moreover, $v$ is linear, i.e., $\forall f, g \in C^0(M)$, $x \in M$ with $x(t) = p$

$$\frac{d}{dt} (f \cdot g)(x(t)) = \left( \frac{d}{dt} f(x(t)) \right) g(x(t)) + f(x(t)) \frac{d}{dt} g(x(t))$$

$$= v(f) \cdot g(p) + f(p) \cdot v(g).$$

This is supposed to motivate:

**Def** A derivation at $p \in M$ is a linear map

$$v : C^0(M) \to \mathbb{R}$$

so that

$$v(fg) = v(f) \cdot g(p) + f(p) \cdot v(g)$$

$$T_p M = \left\{ v : C^0(M) \to \mathbb{R} \mid v \text{ is a derivation} \right\}$$

**Exercise** For $a, b, c \in \mathbb{R}$, $v, w \in T_p M$, $av + bw$ defined by

$$(av + bw)(f) = a \cdot v(f) + b \cdot w(f) \quad \forall f \in C^0(M)$$

In a derivation.

Hence $T_p M$ is a vector space.
Theorem Any derivation is an infinitesimal displacement along some curve. Thus the two definitions agree.<br>
<br>The proof is not hard once we understand a relation between derivations and coordinates.

Let \( M \) be a manifold, \( \varphi = (x_1, \ldots, x_n): U \to \mathbb{R}^n \) a coordinate chart. Then each \( x_i \) is a smooth function on \( U \).

**Def (of \( \frac{\partial}{\partial x_i} \mid_p \, T_p M \))**. For \( f \in C^\infty(M) \) we set

\[
\frac{\partial}{\partial x_i} \mid_p f = \frac{d}{dt} \bigg|_{t=0} (f \circ \varphi')(\varphi(p) + te_i) = \frac{\partial}{\partial u_i} \mid_p (f \circ \varphi')
\]

where \((u_1, \ldots, u_n)\) are coordinates on \( \mathbb{R}^n \).

Exercise \( \frac{\partial}{\partial x_i} \mid_p \) is a derivation.

**Theorem (requires work)** \( \left\{ \frac{\partial}{\partial x_i} \mid_p \right\}_{i=1}^n \) is a basis of \( T_p M \). Hence \( \dim(T_p M) = \dim M \).

Aside For \( f \in C^\infty(M) \) and \( v \in T_p M \), \( v(f) \) only depends on the values of \( f \) near \( p \), as it should. This is another theorem. It follows that if \( U \subset M \) is an open set and \( p \in U \), then \( T_p U = T_p M \).

(Yes, I know, this is a bit fussy.)
Consequence: If $M$ is a manifold, $U \subseteq M$ open, $f \in C^0(U)$, then for each $p \in U$ we have a linear map $df_p : T_p M \to \mathbb{R}$ defined by $df_p(v) := v(f)$ (i.e., $df_p(T^*_p M := (T_p M)^*)$.

This is because $a, b \in \mathbb{R}$ \forall v, w \in T_p M \\& a, b \in \mathbb{R}$

$df_p(av + bw) = a(v + bw)(f) = a v(f) + b w(f) = a df_p(v) + b df_p(w)$

Definition: $df_p$ is the differential of $f$ at $p$.

Now back to coordinates $\psi = (x_1, \ldots, x_n) : U \to \mathbb{R}^n$.

Since $x_1, \ldots, x_n : U \to \mathbb{R}$ are smooth, they define, \forall $p \in U$,

differentials $(dx_i)_p, \ldots, (dx_n)_p \in T^*_p M$.

Lemma: $(dx_1)_p, \ldots, (dx_n)_p$ is a basis of $T^*_p M$ dual to the basis \(\frac{\partial}{\partial x_1}|_p, \ldots, \frac{\partial}{\partial x_n}|_p\).

Proof: $(x_1 \circ \psi)^{-1}(u_1, \ldots, u_n) = (x_1, \ldots, x_n) \circ \psi^{-1}(u_1, \ldots, u_n) = (y \circ \psi^{-1})(u_1, \ldots, u_n) = (u_1, \ldots, u_n)$

$\Rightarrow (x_i \circ \psi^{-1})(u_1, \ldots, u_n) = u_i, \forall u \in \psi(U)$.

$\Rightarrow (dx_j)_p(\frac{\partial}{\partial x_i}|_p) = \frac{\partial}{\partial x_i}|_p(x_j) = \frac{\partial}{\partial u_i}|_{\psi(p)}(x_j \circ \psi^{-1})$

$= \frac{\partial}{\partial u_i}|_{\psi(p)}(u_j) = \delta_{ij}$. \(\square\)

We now recall the tricks with bases and dual bases: If $V$ is a vector space with a basis $\{v_1, \ldots, v_n\}$ and $\{u_1, \ldots, u_n\}$ is the dual basis of $V^*$, then...
\[ \forall w \in V \quad (\#) \quad w = \sum_{i=1}^{n} \langle l_i, w \rangle v_i = \sum \langle l_i | w \rangle v_i \quad (4.6) \]

\[ \forall \eta \in V^* \quad \eta = \sum \eta (v_i) l_i = \sum \langle \eta | v_i \rangle l_i \]

**Consequences**

(1) \( \forall f \in C^0(U) \quad \forall \phi \in \mathcal{U} \)

\[ df_{\phi} = \sum df_{\phi} \left( \frac{\partial}{\partial x_i} \right) \left( dx_i \right)_{\phi} = \sum \frac{\partial f}{\partial x_i} (\phi) \left( dx_i \right)_{\phi} \]

(2) If \( u = (x_1, \ldots, x_n) \), \( \psi = (y_1, \ldots, y_n) : U \to \mathbb{R}^n \) are two coordinate charts then

\[ dy_j = \sum \frac{\partial}{\partial x_i} (y_j) \, dx_i \]

**Note**

\[ \frac{\partial}{\partial x_i} (y_j) = \frac{\partial}{\partial u_j} (y_j \circ \psi^{-1}) = \frac{\partial F_k}{\partial u_j} \]

where

\[ F = (F_1, \ldots, F_n) = \psi \circ \psi^{-1} \]

Similarly,

(3) \[ \frac{\partial}{\partial y_j} = \sum dx_i \left( \frac{\partial}{\partial y_j} \right) \frac{\partial}{\partial x_i} = \sum \left( \frac{\partial x_i}{\partial y_j} \right) \frac{\partial}{\partial x_i} \]