Let $M$ be a manifold, $L : TM \to \mathbb{R}$ a Lagrangian. We assumed: for all $q \in M$, all $v \in T_q M$ the form
\[
d^2L(q,v) : T_q M \times T_q M \to \mathbb{R}
\]
\[
d^2L(q,v)(v_1,v_2) = \frac{\partial^2}{\partial t_1 \partial t_2} L(q, v + tv_1 + tv_2)
\]
is positive definite.

We showed: under this assumption the image $O$ of the Legendre transform
\[\mathcal{L} : TM \to T^*M\]
is an open subset of $T^*M$ and
\[\mathcal{L} : TM \to O\]
a a diffeomorphism. [Pretty soon we'll make a further assumption that $O = T^*M$. This holds for the Lagrangians of the form $L(q,v) = \frac{1}{2} g_{ij}(v,v) + V(v)$ where $g$ is a Riemannian metric on $M$ and $V : M \to \mathbb{R}$ is a smooth function, a potential.]

**Def**. We define the Hamiltonian $H : O \to \mathbb{R}$ associated with $L : TM \to \mathbb{R}$ by
\[H(q,p) = \langle p, L'_{-1}(q,p) \rangle - L(L'_{-1}(q,p))\]
for all $q \in M$, $p \in T_q^*M$.

Here $\langle \cdot, \cdot \rangle : T_q^*M \times T_q M \to \mathbb{R}$ is the canonical pairing.

**Remark**. In coordinates $L(q_1, q_n, v_1, \ldots, v_n) = (q_1 - q_n, \frac{\partial L}{\partial v_1}, \ldots, \frac{\partial L}{\partial v_n})$.

Hence
\[H(q, v) = \sum_{i=1}^n \frac{\partial L}{\partial v_i} \cdot v_i - L(q_1 - q_n, v_1, \ldots, v_n)\]
Recall that there is a vector field $X_L$ on $TM$ associated to $L : TM \rightarrow \mathbb{R}$. The integral curves of $X_L$ satisfy the Euler-Lagrange equations. Recall that we have arrived at these equations by looking at the extremal curves of the action

$$A_L(y) = \int_a^b L(x^i, \dot{x}^i) \, dt.$$ 

On the other hand we have a globally defined tautological 1-form $\alpha \in \Omega^1(T^*M)$, which in coordinates $(q^i, p_i, \rho, \mu) : T^*M \rightarrow \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ is given by

$$\alpha = \sum p_i \, dq_i.$$ 

Then

$$\omega = d\alpha$$

is a globally defined 2-form on $T^*M$. In coordinates

$$\omega = \sum d\rho_i \wedge dq_i.$$ 

Hence $\omega$ is nondegenerate.

Consequently $H : \mathbb{R} \rightarrow \mathbb{R}$ and $\omega$ together define a Hamiltonian vector field

$$X_H = \sum_{i=1}^n \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i}.$$ 

The coordinate-free definition of $X_H$ is

$$\omega(X_H, \cdot) = -dH.$$ 

(Alternatively we could have set $\omega = -d\alpha$ and defined $X_H$ by $\omega(X_H, \cdot) = dH$. Both conventions are used.)
Our goal is to prove:

**Theorem 1** Let $L : TM \to \mathbb{R}$ be a Lagrangian so that the corresponding Legendre transform

$$L_L : TM \to T^* M$$

is a diffeomorphism. Then $\mathcal{L}_q (\xi , \nu) \mathcal{T}q M$

$$(DL_L)_{q,v} \quad X_L(q,v) = X_H (L_L(q,v))$$

To prove the theorem, we need some differential geometry. We start with:

1. **Definition** Let $M, N$ be two manifolds. A map $F : M \to N$ is smooth (equivalently $C^\infty$) if it is smooth in coordinates:

- $U \times M$ is coordinate chart $\psi : U \to \mathbb{R}^m$ (with $x \in U$)
- $V \times N$ is coordinate chart $\psi : V \to \mathbb{R}^n$ (with $f(x) \in V$)

So that $\psi \circ F \circ \psi^{-1} : \psi^{-1}(U \cap \psi^{-1}(V)) \to \psi(V)$ is $C^\infty$.

**Special case:** $N = \mathbb{R}$. Then $\psi : \mathbb{R} \to \mathbb{R}$, $\psi(x) = x$ is a coordinate chart on $N$. Thus $f : M \to \mathbb{R}$ is $C^\infty$ if $U \times M$ is coordinate chart $\psi : U \to \mathbb{R}^m$

$$f \circ \psi : \psi(U) \to \mathbb{R}$$

is $C^\infty$.

**Exercise** If $F : M \to N$ is $C^\infty$ and $f : N \to \mathbb{R}$ is $C^\infty$, then so is $f \circ F : M \to \mathbb{R}$.

**Differentials** If $F : M \to N$ is a smooth map between two manifolds, we define its differential
\[ D F_x : T_x M \rightarrow T_{F(x)} N \quad \text{by} \]
\[ (DF_x(v))f = v(f \circ F) \]
for all \( f \in \mathcal{C}^\infty(N) \), all \( v \in T_x M \).

Exercise Check that \( DF_x \) is linear.

3. Vector fields on manifolds

Definition A vector field \( X \) on a manifold \( M \) assigns to each point \( x \in M \) a vector \( X(x) \in T_x M \) so that the map
\[ X : M \rightarrow TM, \quad x \mapsto X(x) \]
in \( \mathcal{C}^\infty \).

Remark The smoothness of \( X \) amounts to: for any coordinate chart \( \phi = (q_1, \ldots, q_n) : U \rightarrow \mathbb{R}^n \) on \( M \)
\[ X(x) = \sum X_i(x) \frac{\partial}{\partial q_i} \mid _{\phi(x)} \]
where \( X_1, \ldots, X_n : U \rightarrow \mathbb{R} \) are functions.
\( X \) is smooth \( \iff \) \( X_1, \ldots, X_n \) are all smooth.

4. Differential forms on manifolds

Definition A differential 1-form \( \alpha \) on a manifold \( M \) assigns to each point \( x \in M \) a covector \( \alpha_x \in T^*_x M \)
so that the map
\[ \alpha : M \rightarrow T^* M \]
is smooth.

Notation \( \Omega^1(M) = \text{space of differential 1-forms on } M \).
and \( X : Q \to TQ \) is a vector field. The contraction
\[
\tau(X) \omega \in \Omega^1(Q)
\]
\( \tau(X) \omega \) is a 1-form defined by
\[
(\tau(X) \omega)_x(v) = \omega_x(X(x), v)
\]
for all \( x \in Q \), all \( v \in T_x Q \).

It will be useful to look at the linear algebra version of the contraction:

Let \( V \) be a vector space, \( \omega \in \text{Alt}^2(V) \) a skew symmetric bilinear map. Then \( v \in V \) we have a covector \( \omega(v) \in V^* \). It is defined by
\[
(\omega(v) \omega)(u) = \omega(v, u)
\]
for all \( u \in V \).

**Exercise:** The map
\[
\tau : V \times \text{Alt}^2(V) \to V^*, \quad (v, \omega) \mapsto \omega(v)
\]
is bilinear:
\[
\forall a_i, b_j, c_k, d_l \in \mathbb{R}, \quad v_i, v_j \in V, \quad w_i, w_j \in \text{Alt}^2(V)
\]
\[
\tau(a_1 v_1 + a_2 v_2, b_1 w_1 + b_2 w_2) = \sum_{i, j} a_i b_j \tau(v_i) w_j.
\]

**Remark** More generally, for any \( k \geq 1 \) we have a contraction...

\[
\tau : V \times \text{Alt}^k(V) \to \text{Alt}^{k-1}(V)
\]
\[
(\tau(v) \sigma)(u_1, u_{k-1}) = \sigma(v, u_1, \ldots, u_{k-1})
\]
\( \forall v \in V, \quad \sigma \in \text{Alt}^k(V), \quad u_1, \ldots, u_{k-1} \in V \).

**Exercise:** \( \forall l_1, l_2 \in V^* \)
\[
\tau(v)(l_1 \wedge l_2) = l_1(v) l_2 - l_2(v) l_1
\]
Remark. The smoothness of $\alpha : M \to T^* M$ amounts to:

(i) In any coordinates $(q_1, \ldots, q_n) : U \to \mathbb{R}^n$ on $M$

$$\alpha_x = \sum_{i=1}^n \alpha_i \omega^i$$

where $\alpha_i : U \to \mathbb{R}$ are smooth functions.

(ii) For any smooth vector field $X$ on $M$ the function $\alpha(X) : M \to \mathbb{R}, \ x \mapsto \alpha_x(X(x))$

in $C^\infty$.

Definition. A differential 2-form $\omega$ on a manifold $M$ assigns to each point $x \in M$ an alternating bilinear form $\omega_x \in \text{Alt}^2(T_x M)$ so that

(i) For smooth vector fields $X, Y$ on $M$ the function $\omega(X, Y) : M \to \mathbb{R}, \ x \mapsto \omega_x(X(x), Y(x))$

in $C^\infty$.

One can show that (i) is equivalent to:

(ii) For any coordinate chart $(q_1, \ldots, q_n) : U \to \mathbb{R}^n$ on $M$

$$\omega = \sum_{i,j} W_{ij} dq^i \wedge dq^j$$

and $W_{ij} : U \to \mathbb{R}, \ i, j = 1, \ldots, n$ are $C^\infty$.

6) Exterior derivative $d$.

For a smooth function $f : M \to \mathbb{R}$ on a manifold $M$

The 1-form $df \in \Omega^1(M)$ is defined by

$$df_x(v) = v(f)$$

for all $x \in M, \ v \in T_x M$.

In coordinates

$$df = \sum_{i=1}^n \frac{\partial f}{\partial q^i} dq^i.$$
For a 1-form \( \omega \in \Omega^1(M) \), \( d\omega \) is a 2-form.

In coordinates \( d\omega \) is defined by
\[
d ( \sum \omega_i \, dq^i) = \sum \partial \omega_i \wedge dq^i
\]

For a 2-form \( \omega \in \Omega^2(M) \), \( d\omega \) is a 3-form.

In coordinates \( d\omega \) is defined by
\[
d ( \sum \omega_{ij} \, dq^i \wedge dq^j) = \sum \partial \omega_{ij} \wedge dq^i \wedge dq^j
\]
and so on.

In general \( d \) of a \( k \)-form is a \( (k+1) \)-form.

**Definition:** A \( k \)-form \( \gamma \) is **closed** if \( d\gamma = 0 \).

It is **exact** if there is a \( (k-1) \) form \( \xi \) so that \( \gamma = d\xi \).

**Theorem:** \( d\, (d\, \xi) = 0 \).

Hence any exact form is closed.
(we'll won't prove this).

"**Example**" - The canonical 2-form \( \omega \) on the cotangent \( T^*Q \) is exact by definition, hence is closed.

Alternatively, in coordinates
\[
\omega = \sum dp_i \wedge dq^i
\]

\( \Rightarrow \) \( d\omega = 0 \).

Contractions of vector fields and differential 2-forms.

If \( \omega \in \Omega^2(Q) \) is a 2-form on a manifold \( M \).
Exercise \[ \forall l_1, \ldots, l_n \in V^* \quad \forall v \in V \]
\[ \tau(v) (l_1 \wedge \ldots \wedge l_n) = \sum_{c=1}^{k} (-1)^{k-1} l_1 (v) \wedge \ldots \wedge \hat{l}_c \wedge \ldots \wedge l_n \]
where
\[ \hat{l}_c \wedge \ldots \wedge l_n \text{ means that } l_i \text{ was omitted.} \]
(e.g. \[ l_1 \wedge \hat{l}_2 \wedge l_3 = l_1 \wedge l_3 \]
\[ \hat{l}_1 \wedge l_2 \wedge l_3 = l_2 \wedge l_3 \text{ etc.} \]

Proof of theorem 1 on p. 6.3

By definition \( X_H \) is the unique vector field on \( T^*M \) so that
\[ \tau(X_H) \omega = -dH. \]

Therefore, in order to prove that
\[ (DL)_{(q,v)} (X_H(q,v)) = X_H (L(q,v)) \quad \forall q \in M, v \in T_q M \]
it is enough to show that
\[ (DL)_{(q,v)} (X_H(q,v)) \omega_{(q,p)} = -dH_{(q,p)} \]
where \( \omega_{(q,p)} = L(q,v)_\ast \). Since
\[ (DL)_{(q,v)} : T_q M \rightarrow T_p (T^*M) \]
is an isomorphism, (1) is equivalent to
\[ (1) \quad (DL)_{(q,v)} (X_H(q,v)) \omega_{(q,p)} = -dH_{(q,p)} \]
\[ (2) \quad (DL)_{(q,v)} (X_H(q,v)) \omega_{(q,p)} (DL)_{(q,v)} w = -dH_{(q,p)} (DL)_{(q,v)} w \]
for all \( w \in T_q M \).

One more piece of differential geometry that we need: pullback

Let \( F : M \rightarrow N \) be a smooth map. For a \( k \)-form \( \sigma \) on \( N \) we define the \( k \)-form \( F^\ast \sigma \) on \( M \), the pullback of \( \sigma \) by \( F \), by the equation
\[ (F^\ast \sigma)_x (v_1, \ldots, v_k) = \sigma_{F(x)} (DF_x (v_1), \ldots, DF_x (v_k)) \]
for all \( x \in M, v_i \in T_x M \).
Theorem 2.1. For all smooth functions \( f : N \to \mathbb{R} \) and all smooth maps \( F : M \to N \):

\[
F^* d f = d ( f \circ F )
\]

(2) If \( \omega \in \Omega^2(N) \), \( x_i : U \to \mathbb{R}^n \) coordinates on \( N \), then

\[
F^* ( \sum \omega_{ij} \, dx_i \wedge dx_j ) = \sum ( ( \omega_{ij} \circ F ) \cdot d(x_i \circ F) \wedge d(x_j \circ F) )
\]

where \( \omega = \sum \omega_{ij} \, dx_i \wedge dx_j \).

Proof. Omitted.

We now look at the RHS of (2) above.

\[
- d H(q, v) \cdot \left( (dl)_*(q, v), w \right) = - (l^*_c d H)_*(q, v) \cdot (w) = - \left( (l^*_c H)_*(q, v) \right) (w)
\]

(2) Theorem 2.1 above)

On the other hand, the LHS of (2) is

\[
\omega \cdot (dl)_*(q, v) \cdot (dl)_*(q, v) \cdot w = (l^*_c \omega)_*(q, v) \cdot (dl)_*(q, v) \cdot w
\]

Therefore, since \( T(q, v) \cdot (TM) \) is arbitrary, it suffices to show:

(3) \(- d (l^*_c H) = (l^*_c \omega)_*(q, v) \cdot (dl)_*(q, v) \cdot w \)

We next compute in coordinates. Choose coordinates

\( (q_1, \ldots, q_k) : U \to \mathbb{R}^n \) on \( M \). Let

\( (q_1, q_2, \ldots, q_k, v_1, \ldots, v_n) : T^* U \to \mathbb{R}^n \times \mathbb{R}^n \) and

\( (q_1, \ldots, q_k, p_1, \ldots, p_n) : T^* M \to \mathbb{R}^n \times \mathbb{R}^n \)

denote the corresponding coordinates on \( TM \) and \( T^* M \), respectively.

In these coordinates

\[
l_c \cdot \left( q_1, q_2, \ldots, q_k, v_1, \ldots, v_n \right) = \left( q_1, \ldots, q_k, \frac{\partial L}{\partial v_i}, - \frac{\partial L}{\partial q_i} \right)
\]

and

\[
\omega = \sum dp_i \wedge dq_i.
\]

Consequently

\[
d(l^*_c H) = d \left( \langle v, L(q, v) \rangle \right) = d \left( \sum \frac{\partial L}{\partial v_i} \cdot v_i - L(q, v) \right) = \sum \left( \frac{\partial L}{\partial v_i} \cdot dv_i + v_i d \left( \frac{\partial L}{\partial q_i} \right) - \frac{\partial L}{\partial q_i} dq_i \right) - \frac{\partial L}{\partial v_i} \cdot dv_i
\]

\[
= \sum \left( dv_i \cdot d \left( \frac{\partial L}{\partial v_i} \right) - \frac{\partial L}{\partial q_i} dq_i \right)
\]
Recall that \[ x_L = \sum_k \left( \nu_k \frac{\partial}{\partial q_k} + B_k \frac{\partial}{\partial v_k} \right) \]

where \[ B_k = \sum_i \left( \left( \frac{\partial L}{\partial q_i} \right)_k \frac{\partial L}{\partial v_i} - \frac{\partial L}{\partial v_i} \right) \]

- \[ L^* \omega = \sum dq_i \wedge d \left( \frac{\partial L}{\partial v_i} \right) \] by Theorem 2 (2). The formula for \( L^* \omega \) in coordinates and the fact that \( \omega = \Sigma dp_i \wedge dq_i \).

Finally, \(\omega(x_L)\) \[ L^* \omega = \sum_{i,k} \left( \nu_k \frac{\partial}{\partial q_k} + B_k \frac{\partial}{\partial v_k} \right) dq_i \wedge d \left( \frac{\partial L}{\partial v_i} \right) \]

\[ = \sum_{i,k} \left( \nu_k \frac{\partial}{\partial q_k} \left( \frac{\partial L}{\partial v_i} \right) d \left( \frac{\partial L}{\partial v_i} \right) - d \left( \frac{\partial L}{\partial v_i} \right) \right) dq_i + B_k \left( dq_i \left( \frac{\partial}{\partial v_k} \right) \right) \cdot d \left( \frac{\partial L}{\partial v_i} \right) \]

\[ = \sum_{i,k} \left( \nu_k \frac{\partial}{\partial q_k} \left( \frac{\partial L}{\partial v_i} \right) - \nu_k \frac{\partial L}{\partial v_i} \right) dq_i - B_k \left( \frac{\partial}{\partial v_k} \right) \cdot d \left( \frac{\partial L}{\partial v_i} \right) dq_i \]

\[ = \sum_{i,k} \nu_k \frac{\partial}{\partial q_k} \left( \frac{\partial L}{\partial v_i} \right) \sum_{i,k} \left( \nu_k \frac{\partial}{\partial q_k} \left( \frac{\partial L}{\partial v_i} \right) - \nu_k \frac{\partial L}{\partial v_i} \right) dq_i - \frac{1}{2} \sum_{i,k} \left( \frac{\partial}{\partial v_i} \right) \cdot \sum_{j,k} \left( \frac{\partial L}{\partial v_i} \right) dq_i \]

(\text{where we used} \( \sum_k B_k \frac{\partial L}{\partial v_k} \))

\[ = \sum_i \left( \nu_i \frac{\partial}{\partial v_i} - \frac{\partial}{\partial q_i} \right) dq_i = d \left( L^* \omega \right). \]