Let $M$ be a manifold, $L : TM \to \mathbb{R}$ a Lagrangian. The Legendre transform

$$L_L : TM \to \mathbb{T}M$$

is defined as follows: for $q \in M$, $v \in T_q M$,

$$L_L(q, v)$$

is the unique covector in $T_q^* M$ so that

$$\mathcal{L}_L(q, v) \otimes w = \frac{d}{dt}|_{t=0} L(q, v + tw)$$

for any $w \in T_q M$.

Let's see what this means in coordinates.

Let $(q_1, \ldots, q_n) : U \to \mathbb{R}^n$ be coordinates on $M$.

Then $(q_1, \ldots, q_n, v_1, \ldots, v_n)$ are the corresponding coordinates on $TU \subset TM$; recall that for $w \in T_q M$,

$$v_i(w) = (dq_i)_\otimes(w)$$

and $(q_1, \ldots, q_n, v_1, \ldots, v_n)$ the coordinates on $T^*U \subset T^*M$.

For $\eta \in T^*_q M$, $p_\eta(q) = \eta(q \frac{\partial}{\partial q_i})$

$$L(q, v) = L(q_1, \ldots, q_n, v_1, \ldots, v_n)$$

For any $w = (w_1, \ldots, w_n) \in T_q U$,

$$\frac{d}{dt}|_{t=0} L(q, v + tw) = \frac{d}{dt}|_{t=0} L(q_1, \ldots, q_n, v_1 + tw_1, \ldots, v_n + tw_n)$$

$$= \sum_{i=1}^n \frac{\partial L}{\partial v_i}(q, v) w_i = \sum_{i=1}^n \frac{\partial L}{\partial v_i}(q, v) \frac{\partial}{\partial q_i}(w)$$

$$= \mathcal{L}_L(q_1, \ldots, q_n, v_1, \ldots, v_n) = L_L(q_1, \sum v_i \frac{\partial}{\partial q_i}) = (q \sum \frac{\partial L}{\partial v_i}(q, v)(dq_i)_\otimes)$$

$$= (q_1, \ldots, q_n, \frac{\partial L}{\partial v_1}(q, v), \ldots, \frac{\partial L}{\partial v_n}(q, v))$$
If $L: TM \to \mathbb{R}$ is a regular Lagrangian then

Claim: $L_L: TM \to T^* M$ is a local diffeomorphism

Proof: We compute the differential of $L_L$ in coordinates and apply the inverse function theorem.

Since $L_L(q^i, q^j, \dot{q}^i, \dot{q}^j) = (g_{ij}, \frac{\partial L}{\partial q^j}(q, \dot{q}), \ldots, \frac{\partial L}{\partial \dot{q}^j}(q, \dot{q}))$

$$(DL_L)_{(q, \dot{q})} = \left( \begin{array}{c c}
\frac{\partial g_{ij}}{\partial q^k} (q, \dot{q}) & \frac{\partial g_{ij}}{\partial \dot{q}^k} (q, \dot{q}) \\
\frac{\partial L}{\partial q^j} (q, \dot{q}) & \frac{\partial L}{\partial \dot{q}^j} (q, \dot{q})
\end{array} \right) = \left( \begin{array}{c c}
1 & 0 \\
0 & 1
\end{array} \right)$$

$\Rightarrow \det (DL_L(q, \dot{q})) \neq 0 \Rightarrow \det \left( \frac{\partial L}{\partial \dot{q}^j}(q, \dot{q}) \right) \neq 0$

Recall: $L: TM \to \mathbb{R}$ is regular $\iff$ $\det \left( \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}(q, \dot{q}) \right) \neq 0$.

Hence the $(DL_L)(q, \dot{q})$ is invertible for all $(q, \dot{q}) \in TM$.

$\Rightarrow L_L: TM \to T^* M$ is a local diffeomorphism.

Proposition: If $(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}(q, \dot{q}))$ is positive definite for all $(q, \dot{q}) \in TM$, then

$L_L: TM \to T^* M$

is globally 1-1.

Proof: Note that since $q \in M$

$L_L(Tq M) \subseteq T^*q M$

we only need to show that

$L_L: Tq M \to T^*q M$

is 1-1 for every $q \in M$.

Therefore it is enough to prove: Suppose $f: \mathbb{R}^n \to \mathbb{R}$ is a function such that $\nabla \times \nabla^2 f$ is the matrix of second partials $\frac{\partial^2 f}{\partial x_i \partial x_j}(x)$ is positive definite. Then $L_L: \mathbb{R}^n \to (\mathbb{R}^n)^*$, defined by
We first consider the case $n=2$. Then $f: \mathbb{R} \to \mathbb{R}$ is just a function of 1 variable and $L_f(x): \mathbb{R}^2 \to \mathbb{R}$ is the differential $df_x$. In fact
\[
L_f(x)w = \frac{d}{dx} f(x+tw) = f'(x)w ( = df_x(w))
\]
We can think of $L_f: \mathbb{R} \to (\mathbb{R}^2)^*$ as sending $x \to \mathbb{R}$ to the linear functional $f: \mathbb{R} \to \mathbb{R}$,
\[
e(0, f'(x), x) = (f''(x)) \quad [a \times 1 \text{ matrix}]
\]
so "\(\frac{d^2 f}{dx^2}(x)\) is positive definite" means:
\"f''(x) > 0 for all x."

We now identify $L_f(\mathbb{R})^*$ with its slope.

With this identification, $L_f: \mathbb{R} \to \mathbb{R}$, $L_f(x) = f''(x)$.

Now, for $l \in \mathbb{R}$,
\[
l = L_f(x_0) \Rightarrow \frac{d}{dx} (f(x) - l) = 0 \Rightarrow (x_0) = 0
\]
\text{i.e.} \quad $l = L_f(x_0)$ $\iff$ $x_0$ is a critical point of $f_e(x) = f(x) - l \cdot x$

Since
\[
\frac{d^2}{dx^2} f_e = \frac{d}{dx} (f(x) - l) = f''(x) > 0
\]
for all $x$, the function $f_e(x)$ has only one critical point: This point is a global minimum of $f_e(x)$.

It corresponds to the point $x \in \mathbb{R}$ where the tangent line to the graph $y = f(x)$ has slope $l$.

Consequently, \(L_f: \mathbb{R} \to \mathbb{R}^2 \to 1-1\).
Remark: \( f \) need not be onto:

\[ f(x) = e^x. \]

Convince yourself that \( \mathcal{L}(\mathbb{R}) \) = lines with positive slope.

We now tackle the case of \( n > 1 \); i.e. we consider \( f \in \mathcal{C}^0(\mathbb{R}^n) \) with \( d^2 f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \end{pmatrix} > 0 \) ("> 0" here means positive definite. Richard should think of Sylvester's criterion and explain it to Ven, Bryce and Bill).

Lemma: For \( f \in \mathcal{C}^0(\mathbb{R}^n) \) with \( d^2 f(x) > 0 \) there is at most one critical point.

Proof: If \( y, z \in \mathbb{R}^n \) are two critical points of \( f \)

with \( y \neq z \), consider \( g(t) = f(ty + (1-t)z) \in \mathcal{C}^0(\mathbb{R}) \).

By the chain rule,

\[ g'(t) = df(ty + (1-t)z) \cdot (y - z) \]

\[ = \sum_{i=1}^n \left( \frac{\partial f}{\partial x_i}(ty + (1-t)z) y_i - \frac{\partial f}{\partial x_i}(ty + (1-t)z) z_i \right) \]

where \( t = ty + (1-t)z \).

\[ \Rightarrow g''(t) = \sum_{i,j=1}^n \left( \frac{\partial^2 f}{\partial x_i \partial x_j}(ty + (1-t)z) y_i y_j - \frac{\partial^2 f}{\partial x_i \partial x_j}(ty + (1-t)z) z_i z_j \right. \]

\[ + \left. \frac{\partial^2 f}{\partial x_i \partial x_j}(ty + (1-t)z) z_i y_j \right) \]

\[ = \sum_{i,j=1}^n \left( \frac{\partial^2 f}{\partial x_i \partial x_j}(ty + (1-t)z) (y_i - z_i) (y_j - z_j) \right) \]

\[ = (y - z)^T \begin{pmatrix} \frac{\partial^2 f}{\partial x_i \partial x_j}(ty + (1-t)z) \end{pmatrix} (y - z) > 0 \]

since \( y - z \neq 0 \) and \( \begin{pmatrix} \frac{\partial^2 f}{\partial x_i \partial x_j}(ty + (1-t)z) \end{pmatrix} > 0 \).

But \( g'(0) = df(y) \cdot (y - z) = 0 \) since \( df(y) = 0 \)

and \( g'(1) = df(y) \cdot (y - z) = 0 \) since \( df(y) = 0 \).
Since \( g''(t) < 0 \) for all \( t \), \( g(t) \) is strictly increasing. This contradicts \( g'(0) = g'(1) \).

\[ \Rightarrow \quad y = 2. \]

This proves the lemma.

We now argue that \( Lf : \mathbb{R}^n \rightarrow (\mathbb{R}^n)^* \) is 1-1:
\[ l = df(x) \text{ for some } x \in \mathbb{R}^n \]
\[ \Rightarrow \quad l = df(x) \]
\[ \Rightarrow \quad df(x) \cdot 0 \text{ where } f(x) = f(x) - l(x) \]
But \( f(x) \) has only one critical point by lemma.

\[ \Rightarrow \quad Lf \text{ is 1-1}. \]

**Exercise 1** Let \( g : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \) be an inner product, i.e.
(a symmetric positive definite bilinear form.
\[ g(x, y) = x^T A y \] for a positive definite matrix \( A \))
Let \( f(x) = \frac{1}{2} g(x, x) \); the corresponding quadratic form. Show that for any \( x \in \mathbb{R}^n \)
\[ df(x) \in (\mathbb{R}^n)^* \]
in given by \( df(x) w = g(x, w) \) for all \( w \in \mathbb{R}^n \)
Conclude that
\[ Lf : \mathbb{R}^n \rightarrow (\mathbb{R}^n)^* , \quad x \mapsto df(x) \]
a given by
\[ Lf(x) = g(x, \cdot) \]

Explain why it follows that \( Lf \) is a diffeomorphism.
Exercise 2. Consider the Lagrangian $L : \mathbb{R}^n \to \mathbb{R}$ of the form

$$L(q, \dot{q}) = \frac{1}{2} g(q, \dot{q}) - V(q)$$

where $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a positive definite inner product and $V \in C^\infty(\mathbb{R}^n)$ a smooth function.

Show that the Legendre transform

$$L^* : \mathbb{R}^n \to T^* \mathbb{R}^n = \mathbb{R}^n \times (\mathbb{R}^n)^*$$

is given by

$$L^*(t, \nu) = (q, g^{-1}(t, \nu))$$

Exercise 3. Let $(x_1, \ldots, x_n) : U \to \mathbb{R}^n$ and $(y_1, \ldots, y_n) : U \to \mathbb{R}^n$ be two sets of coordinates on a manifold $M$.

Let $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n) : T^* U \to \mathbb{R}^n \times \mathbb{R}^n$ and $(y_1, \ldots, y_n, \zeta_1, \ldots, \zeta_n) : T^* U \to \mathbb{R}^n \times \mathbb{R}^n$ be the corresponding coordinates on $T^* U \times T^* M$.

Show that

$$\sum_{i=1}^n \xi_i \, dx_i = \sum_{i=1}^n \zeta_i \, dy_i$$

Conclude that there is a 1-form $\omega \in \Omega^1(T^* M)$ with

$$\omega|_{T^* U} = \sum \xi_i \, dx_i$$

for any coordinate chart $(x_1, \ldots, x_n)$ on $M$.

[More traditionally $\alpha$ is written as $\sum p_i \, dq_i$.

Note that $\omega = d\alpha$ is a symplectic form:]

$$d\omega = d(d\alpha) = d(\sum p_i \, dq_i) = 0$$

and $\omega = \sum dp_i \wedge dq_i$ is non-degenerate in any local coordinate chart.