Last time:

1) We defined the pull-back $F^* w$ of a $k$-form $w$ by a map $F: \mathbb{R}^k \rightarrow \mathbb{R}^n$.

Recall: it amounts to substitution.
For example,

$$(\cos t, \sin t)^* (x \, dy - y \, dx) =$$
$$\cos t \, d(\sin t) - \sin t \, d(\cos t).$$

2) We defined integration of a 2-form $w = f(x,y) \, dx \, dy$ over a region $D \subseteq \mathbb{R}^2$:

$$\int_D f(x,y) \, dx \, dy := \int_D \int f(x,y) \, dx \, dy.$$ 

Similarly, we define integration of a 1-form $f(t) \, dt$ over an interval $[a, b]$ by

$$\int_{[a,b]} f(t) \, dt := \int_a^b f(t) \, dt \quad \text{for any } f(t).$$
\[
\int \int \cdots \int_{D} F(x_1, x_k) \, dx_1 \cdots dx_k = \\
\int \int \cdots \int_{D} F(x_1, \ldots, x_k) \, dx_1 \cdots dx_k.
\]

3) If we put pull-back and integration of k-forms together, we get integration of 1-forms over curves:

\[\int_{\gamma} \alpha = \int_{\gamma} (\tilde{x})^* \alpha \]

where \( \tilde{x} : [a, b] \to \mathbb{R}^n \) is a parameterization.

2-forms over surfaces:

\[\int_{\Sigma} \omega = \int_{D} X^* \omega \]

where \( X : D \to \Sigma \) is a parameterization and \( D \subseteq \mathbb{R}^2 \) is a domain.

This can be generalized to arbitrary dimensions.
Def. A parameterised $k$-manifold in $\mathbb{R}^n$ is a differentiable map
\[ X: D \to \mathbb{R}^n \]
where $D \subseteq \mathbb{R}^k$ is a region and $X$ is $1$-1 (except possibly on the boundary $\partial D$).

Thus: A $0$-manifold is a point
A $1$-manifold is a curve
A $2$-manifold is a surface.

We can integrate $k$-forms over $k$-manifolds:
\[
\int_X \omega = \int_D X^* \omega
\]
where $\omega$ is a $k$-form in $\mathbb{R}^n$ and $X: D \to \mathbb{R}^n$ a parameterized $k$-manifold.

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Our goal: (generalized) Stokes’s Theorem:

Let $M$ be a $k+1$ manifold with boundary $\partial M$ and $\omega$ a $k$-form. Then
\[
\int_M \omega = \int_{\partial M} \omega.
\]
Recall how $d$ is defined:

If $f: \mathbb{R}^n \to \mathbb{R}$ is a function,

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i + \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} dx_k.$$

If $\omega = \sum F_{i-k} dx_i \wedge dx_k$, a $k$-form, then

$$d\omega = \sum \left( \frac{\partial F_{i-k}}{\partial x_k} \right) dx_i \wedge dx_k.$$

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**Example:**

$M \subseteq \mathbb{R}^2$ a region.

\[ \alpha = P \, dx + Q \, dy \]  a 1-form.

\[ d\alpha = dP \wedge dx + dQ \wedge dy = \left( \frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy \right) \wedge dx + \left( \frac{\partial Q}{\partial y} dx + \frac{\partial Q}{\partial x} dy \right) \wedge dy = \frac{\partial Q}{\partial y} \, dy \wedge dx + \frac{\partial Q}{\partial x} \, dx \wedge dy = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \wedge dy \]

\[ \int_{M} P \, dx + Q \, dy = \iint_{M} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \wedge dy \]

This is [Green's Theorem](https://en.wikipedia.org/wiki/Green%27s_theorem).
Example

\[ \omega = F_1 \, dy \wedge dz + F_2 \, dz \wedge dx + F_3 \, dx \wedge dy, \]

a form in \( \mathbb{R}^3 \) and \( M \subset \mathbb{R}^3 \)
a region with boundary \( \partial M \):

\[
\begin{align*}
d\omega &= dF_1 \wedge dy \wedge dz + dF_2 \wedge dz \wedge dx + dF_3 \wedge dx \wedge dy \\
&= \left( \frac{\partial F_1}{\partial x} \, dx + \frac{\partial F_1}{\partial y} \, dy + \frac{\partial F_1}{\partial z} \, dz \right) \wedge dy \wedge dz + \left( \frac{\partial F_2}{\partial x} \, dx + \frac{\partial F_2}{\partial y} \, dy + \frac{\partial F_2}{\partial z} \, dz \right) \wedge dz \wedge dx + \left( \frac{\partial F_3}{\partial x} \, dx + \frac{\partial F_3}{\partial y} \, dy + \frac{\partial F_3}{\partial z} \, dz \right) \wedge dx \wedge dy \\
&= \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) \, dx \wedge dy \wedge dz.
\end{align*}
\]

So we get

\[
\int \int \int_M \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) \, dx \wedge dy \wedge dz = \nabla \cdot \mathbf{F}
\]

Gauss's theorem.