1 True/false questions. Circle True or False. You don’t need to give reasons for your answers.
   T / F: Every finite set is countable
   T / F: A bijective map \( f : A \to B \) is invertible
   T / F: If \( f : A \to B, \ g : B \to A \) are two functions such that \( g(f(a)) = a \) for all \( a \in A \) then \( f \) is invertible.
   T / F: If \( f : A \to B, \ g : B \to A \) are two injective functions then there is a bijection \( h : A \to B \).
   T / F: There exists a set \( A \) of the same cardinality as the power set \( P(A) \) of \( A \).
   T / F: The set of natural numbers with the usual + and \( \cdot \) forms a ring.
   T / F: The set of rational numbers is countable.
   T / F: One can use Well-ordering principle for natural numbers to prove that the math induction principle is valid.
   T / F: Composition of functions is associative.
   T / F: Cancellation law fails in \( \mathbb{Z}_{10} \).
   T / F: Cancellation law fails in \( \mathbb{Z}_3 \).

2 Prove that \( 2^n < n! \) for all natural numbers \( n \) greater than 3.

3 Prove that \( n < 2^n \) for all \( n \in \mathbb{N} \).

4a (1) Define what it means for \( A \) to be a set with \( n > 0 \) elements.
   (2) Prove that if \( A \) has \( n > 1 \) elements and \( a \in A \) then \( B = A \setminus \{a\} \) has \( n-1 \) elements.
   (3) Prove that if \( A \) has \( n \) elements and \( f : A \to B \) is a bijection then \( B \) also has \( n \) elements.
   (4) Suppose \( A \) and \( B \) are both countably infinite and \( f : A \to B \) is onto. Does this imply that \( f \) is injective? Explain.
   (5) Prove that if \( A \) and \( B \) are both sets with \( n \) elements \( (n \in \mathbb{N}) \) and \( f : A \to B \) is onto then \( f \) is injective.

4b State the Pigeon hole principle. For extra credit: prove the Pigeon hole principle.

5 Let \( f : A \to B \) be a function \( \{U_i\}_{i \in I} \) be a family of subsets of \( B \). Prove that
   - \( f^{-1}(\bigcup_{i \in I} U_i) = \bigcup_{i \in I} f^{-1}(U_i) \);
   - \( f^{-1}(\bigcap_{i \in I} U_i) = \bigcap_{i \in I} f^{-1}(U_i) \).

   Let \( f : X \to Y \) be a function. Define a relation \( \sim \) on \( X \) by
   \[ x \sim x' \iff f(x) = f(x') \, . \]
   - Prove that \( \sim \) is an equivalence relation.
   - What are the equivalence classes of this relation?
   - Denote the set of equivalence classes of \( \sim \) by \( X/\sim \). Prove that the map \( \bar{f} : X/\sim \to Y \) given by
     \[ \bar{f}([x]) := f(x) \]
     is well-defined. Hint: what do you need to check?
   - Prove that \( \bar{f} : X/\sim \to Y \) is injective.

7 Let \( \sim \) be an equivalence relation on a set \( X \), \( C, D \subset X \) two equivalence classes of \( \sim \). Prove that if \( D \cap C \neq \emptyset \) then \( C = D \).

8 Recall that \( \text{Map}(X,Y) \) denotes the set of all functions from a set \( X \) to a set \( Y \). Prove that there does not exist a surjective map from \( X \) to \( \text{Map}(X,\{0,1\}) \).

(continues on the other side)
9 Prove that there is no bijection from \( \mathbb{Q} \) to \( \mathbb{R} \setminus \mathbb{Q} \).

10 Suppose \( X \) is a set, \( \sim \) an equivalence relation and \( X/\sim \) is the set of the equivalence classes.
(a) Prove that the function \( f : X \to X/\sim \), which takes \( x \in X \) to its equivalence class, is onto.
Hint: it’s very short.
(b) Prove that for any \( x, y \in X \)
\[
x \sim y \iff [x] = [y].
\]

11 Prove that \( f(x) = 3x \) defines a bijection from \( \mathbb{R} \) to \( \mathbb{R} \). Does it define a bijection from \( \mathbb{Z} \) to \( \mathbb{Z} \)? If not, what goes wrong? If it does, what’s the inverse?

12 Let \( A \) and \( B \) be two sets. Show that if \( A \) and \( B \) are countable then so are \( A \cup B \) and \( A \cap B \).

13 Is the set
\[
C = \{(x, y) \in \mathbb{R}^2 \mid x^6 + y^4 = 1\}
\]
the graph of a function from \( \mathbb{R} \) to \( \mathbb{R} \)? Explain.

14 Prove that composition of functions is associative.

15 Prove that if \( f : A \to B \) and \( g : B \to C \) are both onto then so is their composite \( g \circ f : A \to C \).