Last time: We fix a vector space $V$ over a field $F$.

A subset $\{v_1, \ldots, v_m\} \subset V$ spans $V$ if $\forall u \in V$ \exists $\alpha_i \in F$ st.

$u = \sum_{i=1}^{m} \alpha_i v_i$.

A subset $\{v_1, \ldots, v_m\} \subset V$ is linearly independent if

$\sum_{i=1}^{m} \alpha_i v_i = 0 \Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$.

A subset $\{v_1, \ldots, v_m\} \subset V$ is linearly dependent if $\exists \alpha_i \neq 0$ not all zero st. $\sum_{i=1}^{m} \alpha_i v_i = 0$.

We've seen: $\{v_1, \ldots, v_m\}$ is a linearly dependent set $\iff$ there is a linear combination of the rest of $v_i$'s.

We can do better:

Lemma 2.1.15

Suppose $\{v_1, \ldots, v_m\} \subset V$ is a linearly dependent set of non-zero vectors. Then $\exists k, 1 \leq k \leq m$ st. $v_k$ is a linear combination of $v_1, \ldots, v_{k-1}$.

Proof:

Since $v_k \neq 0$, $\{v_1, \ldots, v_{k-1}\}$ is linearly independent.

Let $k$ be the smallest integer st. $\{v_1, \ldots, v_{k-1}\}$ is linearly independent and $\{v_1, \ldots, v_k\}$ is linearly dependent.

Then $\exists$ scalars $\alpha_1, \ldots, \alpha_k$ s.t. $\alpha_1 v_1 + \cdots + \alpha_k v_k = 0$.

And $\alpha_1, \ldots, \alpha_k$ are not all zero. Note that we cannot have $\alpha_k = 0$: This says $v_1, \ldots, v_{k-1}$ are linearly dependent.

$\Rightarrow \alpha_k \neq 0$. $\Rightarrow v_k = -\alpha_k^{-1} (\alpha_1 v_1 + \cdots + \alpha_{k-1} v_{k-1})$.

Definition 2.1.16

A subset $\{v_1, \ldots, v_m\}$ of a vector space $V$ over $F$ is a basis if

1. $\{v_1, \ldots, v_m\}$ is linearly independent
2. $\{v_1, \ldots, v_m\}$ spans $V$.

Example:

$V = F^n \{e_1, \ldots, e_n\}$ is a basis, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$

Reason:

$(x_1, \ldots, x_n) \in F^n \Rightarrow (x_1, \ldots, x_n) = x_1 e_1 + \cdots + x_n e_n \Rightarrow \text{basis}$ spans.

$\sum x_i e_i = 0 \Rightarrow (d_1, \ldots, d_n) = (0, \ldots, 0) \Rightarrow d_1 = d_2 = \cdots = d_n = 0$.
Remark: \( \{ v_1, \ldots, v_m \} \) is a basis of \( V \) if \( V = \text{span}(V) \) and \( \{ v_1, \ldots, v_m \} \) is linearly independent.

\[ u = \sum_{i=1}^m a_i v_i \]

Reason: \( u \in V \) for \( a_1, \ldots, a_m \in F \) so \( u = \sum_{i=1}^m a_i v_i \) spans \( V \).

If \( \{ v_1, \ldots, v_m \} \) is linear independent and
\[ \sum_{i=1}^m a_i v_i = 0 \]

Then \( \sum_{i=1}^m (a_i - b_i) v_i = 0 \) implies \( a_i = b_i \) for all \( i \).

Conversely, suppose \( \sum_{i=1}^m a_i v_i = \sum_{i=1}^m b_i v_i \) then \( a_i = b_i \) for all \( i \)
and \( \sum_{i=1}^m a_i v_i = 0 \). Then \( \sum_{i=1}^m x_i v_i = 0 \) implies \( x_i = 0 \) for all \( i \).

Thus \( \{ v_1, \ldots, v_m \} \) is linear independent.

\[ \sum_{i=1}^m a_i v_i = 0 \]

Definition: A vector space \( V \) is \textit{finite dimensional} over \( k \) if there is a finite set \( \{ v_1, \ldots, v_m \} \) that spans \( V \).

Example: \( F^n \) is finite dimensional over \( F \).

\( F[x] \) is finite dimensional over \( F \).

\( F[x] \) is not finite dimensional.

Reason: Suppose \( p_1, \ldots, p_m \in F[x] \) span \( F[x] \).

Let \( d = \max \{ \deg p_i \mid 1 \leq i \leq m \} \).

Then \( \exists a_1, \ldots, a_m \in F \) such that \( \sum_{i=1}^m a_i p_i = 0 \).

Since \( \deg(\text{RHS}) \leq \max(\deg p_i) \), \( \deg \text{RHS} = d \).

Then \( a^d = \deg(\text{LHS}) = d + 1 \).

If \( V \) is a vector space over a field \( F \) and \( V = \{ \theta \} \), then \( \theta \) spans \( V \). \( V = \{ \theta \} \) is finite dimensional.

One writes \( 0 \) for \( \{ \theta \} \),
This is the \textit{zero} vector space over \( F \).
Lemma 2.1.21. Any non-zero finite dimensional vector space $V$ over a field $F$ has a basis.

Proof. Since $V$ is finite dimensional, $Fv_1, \ldots, Fv_m \in V$ sat. $\{v_1, \ldots, v_m\}$ spans $V$.

If one of the $v_i$'s is $0$, discard it; the rest of the vectors still span $V$. We may assume $v_j \neq 0 \forall j$.

By 2.1.15, $\exists k \geq 1$ s.t. $\{v_1, \ldots, v_{k-1}\}$ are linearly independent and $v_k$ is a linear combination of $v_1, \ldots, v_{k-1}$.

Then $\{v_1, v_2, \ldots, v_k, \ldots, v_m\}$ still spans $V$:

$v_k = \sum_{j=1}^{k-1} \beta_j v_j$ for some $\beta_1, \ldots, \beta_{k-1}$.

If $u = v_1 v_i + \ldots + a_k v_k + a_{k+1} v_{k+1} + \ldots + a_m v_m$, Then

$u = a_1 v_i + \ldots + a_{k-1} v_{k-1} + a_k \left( \sum_{j=1}^{k-1} \beta_j v_j \right) + a_{k+1} v_{k+1} + \ldots + a_m v_m$.

Continue the process. After finitely many steps (perhaps none) we're left with a linearly independent spanning set.

Aside. By convention if $V = 0$, $V$ has an empty basis.

Recall $F^n$ has a basis with $n$ elements: $e_1, \ldots, e_n$.

We'd like to define the dimension of $F^n$ to be $n$, which is the number of elements in a basis of $F^n$.

Issue: Suppose $v_1, \ldots, v_n$ is another basis of $F^n$.

Why is $n = n$?

We need

Lemma 2.1.20 (exchange lemma).

Suppose $V$ is a vector space over a field $F$.

$F v_1, \ldots, F v_m$ spans $V$, $\{v_1, \ldots, v_n\}$ linearly independent.

Then $n \leq m$.

(proof next time)
**Theorem 2.1.22** Suppose \( v_1, \ldots, v_n \) and \( w_1, \ldots, w_m \) are two bases of a vector space \( V \) over \( F \). Then \( n = m \).

**Proof.** Since \( \{v_1, \ldots, v_n\} \) is a basis, it’s lin. independent.

Since \( \{w_1, \ldots, w_m\} \) is a basis, it spans \( V \).

By 2.1.20, \( n \leq m \).

Similarly, \( m \leq n \). Therefore \( n = m \). \( \square \)

We need bases to define dimensions of vector spaces.
But bases are good for other things too.

**Example** Suppose \( T: V \to W \) is a linear map

and \( \{v_1, \ldots, v_n\} \) is a basis of \( V \). Then \( T \) is uniquely determined by \( T(v_1), \ldots, T(v_n) \).

**Reason:** Any \( u \in V \) has \( u = \sum a_i v_i \).

Since \( T \) is linear, \( T(u) = T(\sum a_i v_i) = T(\sum a_i v_i) + \ldots + T(a_n v_n) = a_1 T(v_1) + \ldots + a_n T(v_n) \).

Conversely, given a basis \( \{v_1, \ldots, v_n\} \) of \( V \) and \( w_1, \ldots, w_m \in W \)

\( \exists! T: V \to W \) linear with \( T(v_i) = w_i \). (\( w_1, w_m \) need not be all distinct!)

**Reason:** Given \( u = \sum a_i v_i \), define

\( T(u) = \sum a_i w_i \).

Since \( v_i = 0 \cdot v_1 + \ldots + 0 \cdot v_{i-1} + 1 \cdot v_i + 0 \cdot v_{i+1} + \ldots + 0 \cdot v_n \), \( T(v_i) = w_i \).

Moreover, \( T \) is linear. For example, if \( u' = \sum b_i v_i \),

\( T(u' + u) = T(\sum a_i v_i + \sum b_i v_i) = T(\sum (a_i + b_i) v_i) = T(u) + T(u') \).