Recall \[ B = \left\{ \{a_n\} \in \mathbb{Q}^\mathbb{N} \mid \forall x, y \in \mathbb{Q}, \exists n, m > N \text{ such that } |a_n - b_m| < \epsilon \right\} \]

Then \[ |a_n - a_m| < \epsilon \]

\[ I = \{ a \in \mathbb{Q} | \text{an} \to 0 \} \]

We proved \( I \) is an ideal in the ring \( B \).

\[ IR = B/I \]

\( IR \) is a commutative ring.

Book denotes elements of \( IR \) by \( \lfloor a \rfloor \).

\[ \lfloor a \rfloor = \text{equivalence class of Cauchy sequences in } B \]

\[ = \{ a_i + x_i | x_i \in I, y = a_i + x_i \} \]

\( f \colon \mathbb{Q} \to IR, f(r) \mapsto \text{class of constant sequence } r \)

We've seen \( f \colon \mathbb{Q} \to IR, f \mapsto \text{class of constant sequence } r \) is an injective ring homomorphism.

We identify \( r \in \mathbb{Q} \) with \( \lfloor r \rfloor \in IR \).

\( IR \) is a commutative ring with \( 1 = \lfloor 1 \rfloor \).

We also proved:

\[ \text{Lemma 20.2} \] Suppose \( \text{an} \in \mathbb{Q}^\mathbb{N} \) and \( a_n \to 0 \). Then \( \exists M \in \mathbb{Q}, d \in \mathbb{Q} \)

with \( d > 0 \) such that either \( a_n > d \) \( \forall n > M \)

or \( a_n < -d \) \( \forall n > M \).

**Theorem 3.5.18** \( IR = B/I \) is a field.

\( \text{Proof:} \) we need to show: \( \forall x \in IR, x \neq 0, f(x)IR \leq x, y = 1 \)

\( x \neq 0 \) means: \( x = \lfloor a_n \rfloor + I \) and \( x \neq 0 + I, = I \), ie.

\[ 4a_n \notin I, \text{ ie. } a_n \to 0. \]

By 20.2 \( \exists M, d > 0 \) such for \( |a_n| > d \) for \( n > M \).

Let \( b_n = \begin{cases} \frac{1}{n} & n \leq M \\ 1/a_n & n > M \end{cases} \)

Then by construction \( b_n \cdot a_n = 1 \) for \( n > M \)

\[ \Rightarrow a_n \cdot b_n - 1 \to 0. \]

\[ \Rightarrow 1/b_n \to 1 \in I \]

\[ \Rightarrow (a_n + I) \cdot (b_n + I) = a_n \cdot b_n + I = 1 + I \]

ie. \( a_n \lfloor b_n \rfloor = 1 \in IR. \)

Oops. We forgot to check that \( 1/b_n \in IR \).
Claim 4.6 \( n \in I \).

Reason: For \( n, m > M \),
\[
|b_n - b_m| = \left| \frac{1}{a_n} - \frac{1}{a_m} \right| = \frac{|a_m - a_n|}{|a_n||a_m|} < \frac{1}{d^2} |a_n - a_m|.
\]

Since \( a_n \) is Cauchy, \( \forall \varepsilon > 0 \) \( \exists N \) st. \( |a_n - a_m| < d^2 \varepsilon \) for \( n, m > N \).

Then for \( n, m > \max(N, N') \)
\[
|b_n - b_m| < \frac{1}{d^2} \cdot d^2 \varepsilon = \varepsilon.
\]

Lemma 2.1 \( R = E/I \) is an ordered field.

Recall, given a field \( F \) and \( P \subset F \setminus \{0\} \) such that
1. \( \forall x, y \in P \), either \( x \notin P \) or \( -x \notin P \) and
2. \( \forall x, y \in P \), \( x \cdot y \in P \) and \( x + y \in P \)
we have a relation \( < \) on \( F \):
\[
a < b \iff b - a \in P.
\]
(\( F, < \) ) is then an ordered field.

Proof of 2.1. Let \( P = \{ a_n + I \mid \exists M, d \in \mathbb{R}, a_n > d \text{ for } n > M \} \).

(Note \( \frac{1}{d} + I \notin P \) even though \( \frac{1}{n} \to 0 \) as \( n \to \infty \).)

We need to check:
1. \( P \) is well-defined,
2. \( P \subset \mathbb{R} \setminus \{0\} = \mathbb{R} \setminus \frac{1}{n} + \mathbb{I} \) and satisfies (1), (2).

Suppose \( (a_n) + I = (a_n') + I \) and \( \exists M, d > 0 \) with \( a_n > d \) for \( n > M \). Need to show:\n\[
\exists M', d' \text{ st. } a_n' > d' \text{ for } n > M'.
\]
Since \( (a_n) + I = (a_n') + I \), \( a_n - a_n' \to 0 \). \( \Rightarrow \)
\[
\exists N \text{ st. for } n > N, \quad |a_n - a_n'| < \frac{d}{2}.
\]
If \( n > \max(N, M) \) then
\[a_n > d, \quad |a_n - a_n'| < \frac{d}{2}. \quad \Rightarrow \quad a_n' > \frac{d}{2}.\]

**Claim 2.1.3.**

\[\forall x \in \mathbb{R}, x > 0 \text{ either } x \in P \text{ or } -x \in P.\]

**Reason:** Suppose \( x > \{a_n + i \neq \mathbb{I} \}. \) Then \( x > 0. \) By 2.0.2

\[\exists M, d > 0 \text{ st. either } a_n > d \text{ for } n > M \text{ (then } a_n + i \in P)\]

or \( a_n < -d \text{ for } n > M \text{ (then } -a_n + i \in P)\)

**Claim:** \( P \) is closed under \( + \) and \( \cdot.\)

**Reason:** If \( \{a_n + i \}, \{b_n + i \} \in P \) then \( \exists M, d_1, d_2 > 0\)

\[\iff a_n > d_1 \land b_n > d_2 \text{ for } n > M.\]

\[\Rightarrow a_n \cdot b_n > d_1 \cdot d_2 > 0 \text{ for } n > M \text{ and}\]

\[a_n + b_n > d_1 + d_2 > 0 \text{ for } n > M.\]

**Final goal:** Thm 3.5.22. Any subset \( A \subset \mathbb{R} \) bounded above has a least upper bound.

**Lemma 21.2**

\( \mathbb{Q} \) is an Archimedean ordered field: \( \forall \frac{a}{b} \in \mathbb{Q}, (a, b \in \mathbb{Z})\)

\[\exists N \in \mathbb{Z} \text{ with } \frac{a}{b} < N.\]

**Proof:** We may assume \( b > 0. \) By the division algorithm

\[\exists q, r \in \mathbb{Z} \text{ s.t. } a = q \cdot b + r \text{ and } 0 \leq r < b.\]

Then \( a = q \cdot b + r \leq q \cdot b + b = (q + 1) \cdot b\)

\[\Rightarrow \frac{a}{b} < q + 1 \in \mathbb{Z}.\]

**Thm 3.5.21**

\( \mathbb{R} \) is an Archimedean ordered field: \( \forall x \in \mathbb{R} \)

\[\exists N \in \mathbb{Z} \text{ st. } x < N.\]

**Proof:** \( x = \{a_n\} = \{4a_n\} + i \) for some Cauchy sequence \( \{a_n\} \) of rational numbers.

Since \( \{a_n\} \) is Cauchy, it's bounded by some \( \mathbb{R} \).
Since \( \mathbb{R} \) is Archimedean, \( \exists N \in \mathbb{N} \) with \( r \leq N \).

\[ \Rightarrow \forall n \in \mathbb{N}, \quad a_n \leq |a_n| \leq r \leq N. \]

\[ \Rightarrow N - a_n > 0 \quad \forall n. \]

\[ (N + 1) - (2a_n + 1) = (N - a_n) + 1 \in \mathbb{P}. \]

\[ \Rightarrow N + 1 > 2a_n + 1 = x. \]

Note: We proved several weeks ago: if \( F \) is an ordered field with least upper bound property then \( F \) is Archimedean. Here we first prove that \( \mathbb{R}/\mathbb{Z} \) is Archimedean and use this fact to prove that \( \mathbb{R}/\mathbb{Z} \) has least upper bound property.