Last time. A nonempty subset $I$ of a commutative ring $R$ is an ideal if
1. $\forall x,y \in I, \ x - y \in I$
2. $\forall r \in R, \forall x \in I, \ rx \in I$.

Remarks (i) An ideal $I \subseteq R$ is a subring: $0 \in I$, $I$ is closed under $+$ and $\cdot$, etc.

Note. $\mathbb{Z} \subseteq \mathbb{Q}$ is a subring, but not an ideal: $\frac{1}{2} \cdot 3 \notin \mathbb{Z}$.

(ii) $\{0\}$ and $R$ are ideals in $R$.

2. An ideal $I$ in a ring $R$ defines an equivalence relation $\sim$:

\[ a \sim b \iff a - b \in I. \]

The equivalence classes of $\sim$ are of the form

\[ [r] := \{ r + x \mid x \in I \} = r + I. \]

(iii) Notation $R/I = \{ r + I \mid r \in R \}$ is the set of equivalence classes of $\sim$.

3. Thm 19.2. Let $I$ be an ideal in a commutative ring $R$. Then the set $R/I$ is a ring with $+$ and $\cdot$ given by

\[ (r + I) + (q + I) = (r + q) + I \]
\[ (r + I) \cdot (q + I) = rq + I \]

Hence $\pi : R \to R/I$, $\pi(r) = r + I$ is a ring homomorphism.

Didn't prove:

Lemma 19.3. The set $I = \{ \sum_{n=0}^{\infty} a_n x^n \in \mathbb{C} \mid a_0 + \cdots + a_n \in I \}$ is an ideal in the ring $\mathbb{C}$ of Cauchy sequences in $\mathbb{Q}$. Hence $R/I = \mathbb{C}/I$ is a ring

and $\pi : \mathbb{C} \to R$, $\pi(\sum_{n=0}^{\infty} a_n x^n) = a_0 + I$

is a ring homomorphism.

Note. The book writes $La.7$ for $a_0 + I$.

Proof of 19.3. We need to check two things.
(2) \( \forall \lambda_n, \lambda_n \in I, \quad \| \lambda_n \| - \| \lambda_n \| \in I \).

Since \( \lambda_n \to 0 \) and \( b_n \to 0 \), \( \forall \varepsilon > 0 \) \( \exists N \) such that \( n \geq N \) then
\[
|\lambda_n - b_n| < |\lambda_n| + |b_n| < \varepsilon / 2,
\]
This proves (1).

(2) Fix \( \varepsilon > 0 \). Since \( x_n \in C \), it's bounded: \( \exists M > 0 \)
sl. \( |x_n| < M \forall n \). Since \( \lambda_n \to 0 \), \( \exists N \) such that \( n \geq N \)
then \( |\lambda_n| < \varepsilon / M \). Hence for \( n \geq N \)
\[
|\lambda_n x_n| = |\lambda_n| \cdot |x_n| < \frac{\varepsilon}{M} \cdot M = \varepsilon.
\]
\[\Rightarrow \lambda_n \cdot x_n \to 0.\] This proves (2).

Remark: We have a injective ring homomorphism
\[ \mathbb{Q} \to \mathbb{R}, \quad r \mapsto r + I \] where \( r \) is
the constant sequence \( a_n = r \forall n \).

It's a homomorphism since
\[
(r + r') + I = (r + I) + (r' + I) = (r + I) + (r' + I)
\]
and
\[
(r + r') + I = (r + I) \cdot (r' + I).
\]
It's injective because
\[
r + I = r' + I \Leftrightarrow r - r' \in I \Rightarrow \text{the constant sequence}
\]
\( a_n = r - r' \forall n \) converges to 0 \( \Rightarrow r - r' = 0.\)

Next goal: Theorem 3.5.18 \( R = \mathbb{R} / I \) is a field.

We first prove

Lemma 20.1: Suppose \( a, b, c \in \mathbb{Q}, \quad |a| > c > 0 \) and
\[|a - b| < c/2.\] Then
(1) if \( a > c \), then \( b > c/2 \),
(2) if \( a < -c \), then \( b < -c/2 \).
Proof of (1) (proof of (2) is similar)

\[ |a-b| < \varepsilon/2 \Rightarrow a-b < \varepsilon/2 \Rightarrow b > a - \varepsilon/2 \geq c - \varepsilon/2 = \varepsilon/2. \]

Lemma 20.2 (compare with 3.5.12) Suppose \((a_n) \in \mathbb{R}\) is a Cauchy sequence and \(a_n \not\to 0\), i.e., \(\exists \varepsilon > 0\) such that \(\forall M \in \mathbb{N}, \exists n > M\) such that \(a_n \notin (-\varepsilon, \varepsilon)\).

Proof 1. Since \(a_n \not\to 0\), it's not true that \(\exists N \in \mathbb{N}\) so that if \(n \geq N\) then \(|a_n - 0| < \varepsilon\). Therefore \(\exists \varepsilon_0 > 0\) so that for any \(N \in \mathbb{N}\) \(\exists n \geq N\) \(n \geq N\) and \(|a_n| \geq \varepsilon_0\).

1. Since \((a_n)\) is Cauchy, \(\exists M\) so that if \(k, l \geq M\) then \(|a_k - a_l| < \varepsilon_0/2\).

By (1), \(\exists n_0 \geq M\) with \(|a_{n_0}| \geq \varepsilon_0\).

By 20.1 \(\forall k > M\), either \(a_k > \varepsilon_0/2\) (if \(a_{n_0} \geq \varepsilon_0\)) or \(a_k < -\varepsilon_0/2\) (if \(a_{n_0} \leq \varepsilon_0\)).

Take \(d = \varepsilon_0/2\).

Proof of 3.5.18 Suppose \(\frac{1}{a_n} + 1 \neq 0 + 1\), i.e., \(\exists \varepsilon > 0\) and \(a_n \not\to 0\).

By 20.2, \(\forall M, \exists d > 0\) such that \(|a_n| > d\) for \(n \geq M\).

Let \(b_n = \frac{1}{d} \frac{1}{a_n}\) for \(n \geq M\).

Then for \(n, m \geq M\),

\[ |b_n - b_m| = |\frac{1}{a_n} - \frac{1}{a_m}| < \frac{1}{d^2} \frac{|a_n - a_m|}{|a_n||a_m|} < \frac{1}{d^2} |a_n - a_m|. \]

Since \((\frac{1}{a_n})\) is Cauchy \(\forall \varepsilon > 0\), \(\exists N \in \mathbb{N}\) \(\forall n, m \geq N \Rightarrow |a_n - a_m| < d^2 \varepsilon\).

Then for \(n, m \geq \max(N, M)\)

\[ |b_n - b_m| < \frac{1}{d^2} |a_n - a_m| < \frac{1}{d^2} d^2 \varepsilon = \varepsilon. \]

\(\Rightarrow \) \(b_n\) is Cauchy.

By construction \(a_n \cdot b_n = 1\) for \(n \geq M\) \(\Rightarrow a_n \cdot b_{n-1} \to 0\) as \(n \to \infty\).
\[ (q_n + I) (q_n + I) = q_n + I \]

\[ \therefore \quad \mathbb{R} = \mathbb{F}/I \in \text{a field} \]

**Order for \( \mathbb{R} \).**

Recall defining \( < \) on a field \( \mathbb{F} \) is equivalent to singling out \( P \subseteq \mathbb{F}/I \) so that:

1. \( \forall x \in P, x \neq 0 \) either \( x \in P \) or \( -x \in P \)
2. \( \forall a, b \in P \quad a + b, a \cdot b \in P \)

We then define \( x < y \in \mathbb{F} \iff y - x \in P \).

We define \( P \subseteq \mathbb{R} = \mathbb{F}/I \) by

\[ P = \{ q_n + I \mid \exists M, d > 0 \text{ st } a_n > d \text{ for } n > M \} \]

**Claim 1** \( P \) is well-defined.

**Proof** Suppose \( q_n + I = q_{n'} + I \) and \( \exists M, d > 0 \) with \( a_n > d \) for \( n > M \).

Need to show: \( \exists M', d' > 0 \) st \( a_n > d' \) for \( n > M' \).

Since \( a_n - d_n \to 0 \), \( \exists N \) st. if \( n > N \) then \( |a_n - d_n| < d/2 \).

Then for \( n > \max(N, M) \)

\[ a_n - d/2 > 0 \text{ by 20.1} \]

\( P \) is well-defined.

**Claim 2** \( \forall (a_n + I) \in \mathbb{R}, \& (a_n + I \neq 0 + I) \), either \( a_n + I \in P \)

or \( -a_n + I \in P \).

**Proof** Since \( (a_n + I \neq 0 + I), a_n \to 0 \). By 20.2

\( \exists M, d > 0 \) st. either \( a_n > d \text{ for } n > M \) or \( a_n < -d \text{ for } n > M \).

\( \Rightarrow \) either \( a_n + I \in P \) or \( -a_n + I \in P \).

**Claim 3** if \( (a_n + I, b_n + I) \in P \) then so are

\[ (qa_n + I, \& b_n + I) + I \text{ and } q(a_n + I \cdot b_n + I) + I \]