Last time - Reviewed notion of a ring and ring homomorphism.

- Proved a criterion for a subset \( S \) of a ring \( R \) to be a subring.
- For any ring \( R \), the set \( R^n = \{ r_1, r_2, \ldots, r_n \mid r_i \in R \} \) is a ring.
- Almost proved: the set \( B \) of Cauchy sequences in \( \mathbb{Q} \) is a subring of \( \mathbb{Q}^\infty \).

We proved:
- If \( \{a_n\} \) is Cauchy then \( \{a_n\} \) is bounded.
- If \( \{a_n\}, \{b_n\} \) Cauchy, then so is \( \{a_n + b_n\} \).

Left to prove:

If \( \{a_n\}, \{b_n\} \) are Cauchy sequences in \( \mathbb{Q} \) then so is \( \{a_n - b_n\} \).

Proof

Let \( \varepsilon > 0 \) be fixed. Choose \( N_1 \) such that for all \( n, m \geq N_1 \),

\[ |a_n - a_m| < \frac{\varepsilon}{2} \]

and choose \( N_2 \) such that for all \( n, m \geq N_2 \),

\[ |b_n - b_m| < \frac{\varepsilon}{2} \]

Then for \( n, m \geq \max(N_1, N_2) \),

\[ |(a_n - b_n) - (a_m - b_m)| \\leq |a_n - a_m| + |b_m - b_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

Note: The constant sequence \( \{1\} \), \( \{a + 1\} \), \( \{a_n\} \) are Cauchy (as in any constant sequence in \( \mathbb{Q} \)).

\( \Rightarrow \) \( B \) is a commutative ring with \( 1 \).

Next we need to divide out by the relation \( \sim \) where

\[ \langle a_n \rangle \sim \langle b_n \rangle \iff a_n - b_n \to 0. \]

We prove that \( B/\sim \) is again a ring.

Definition: Let \( R \) be a (commutative) ring (with 1).

A subset \( \mathfrak{I} \subseteq R \) is an ideal if

1. \( \forall a, b \in \mathfrak{I}, \quad a - b \in \mathfrak{I} \)
2. \( \forall r \in R, \forall a \in \mathfrak{I}, \quad ra \in \mathfrak{I} \).
Theorem 19.2
Let $R$ be a commutative ring, $I \subseteq R$ an ideal. Then $R/I$ is a ring and so that
\[ \pi : R \to R/I, \quad \pi(r) = r + I \]
\[ \text{is a ring homomorphism.} \]

Proof
We define $+$ and $-$ on $R/I$ by
\[ (r + I) + (q + I) := (r + q) + I \]
\[ (r + I) \cdot (q + I) = r q + I. \]

Need to check:
(i) if $r + I = r' + I$, $q + I = q' + I$ then
\[ (r + q) + I = (r' + q') + I \]
(ii) $r q + I = r' q' + I$.

Now $r + I = r' + I \Rightarrow r = r' + x$ for some $x \in I$
$q + I = q' + I \Rightarrow q = q' + y$ for some $y \in I$
\[ \Rightarrow (r + q) - (r' + q') = (r - r') + (q - q') = x + y \in I \]
\[ \Rightarrow (r + q) + I = (r' + q') + I. \]

Similarly
\[ r q - r' q' = (r' + x) \cdot (q' + y) - r' q' \]
\[ = r' q' + x q' + r' y + x y - r' q' \]
\[ = x q' + r' y + x y \in I. \]

Lemma 19.2
The set $I = \{ s \in \mathbb{Q} \mid \text{an} a_n \to 0 \}$ is an ideal in the ring $\mathbb{C}$ of Cauchy sequences. Hence
\[ \mathbb{C} / I = \mathbb{C} / \sim \text{ is a ring and } \pi : \mathbb{C} \to \mathbb{C} / I \]
\[ \text{is a surjective ring homomorphism.} \]

Proof
We need to check:
(i) if $s \in I$, $s = a_n \sim I \Rightarrow (s + I) = s + I \in I$
(ii) if $s \cdot s' \in \mathbb{C}$, then $s' \in I$, $s \cdot s' \in I.$
Example. \( R = \mathbb{Z}, \ n \mathbb{Z} = \{ m \in \mathbb{Z} \mid n \mid m \} \)

\[ \equiv \{ n k \mid k \in \mathbb{Z} \} \]

in an ideal:

\[ \forall k_1, k_2 \in \mathbb{Z}, \ nk_1 - nk_2 = n (k_1 - k_2) \in n \mathbb{Z} \]

\[ \forall m \in \mathbb{Z}, \ n \left( nk + n \mathbb{Z} \right) = m(hk) = n(mk) \in n \mathbb{Z} \]

Lemma 19. Let \( R \) be a ring, \( I \subseteq R \) an ideal.

Then \( \sim \) the relation \( \sim \) defined by \( x \sim y \iff x - y \in I \)

is an equivalence relation.

Proof. Note that since \( I \neq \emptyset \), \( \exists x \in I \) \( \iff 0 = x - x \in I \)

\[ = \forall r \in R, \ r - r = 0 \in I \Rightarrow r \sim r. \]

(2) \( \forall a \in I, \ (I + x) = 0 - x \in I \)

Suppose \( r \sim r' \). Then \( r - r' \in I \Rightarrow -(r - r') \in I \)

But \( -(r - r') = r' - r \Rightarrow r' \sim r \)

(3) \( \forall x, y \in I \), \( x + y = x - (-y) \in I \)

Now if \( r \sim r' \) and \( r' \sim r'' \)

Then \( r - r', r' - r'' \in I \Rightarrow r - r'' = (r - r') + (r' - r'') \in I \)

\[ \Rightarrow r \sim r'' \]

\[ \therefore \sim \text{ is an equivalence relation.} \]

Ex. If \( R = \mathbb{Z} \), \( I = n \mathbb{Z} \), \( m \sim m' \iff m - m' \in n \mathbb{Z} \)

\[ \equiv n \mid m - m' \]

The set of equivalence classes \( \mathbb{Z}/\sim \) in \( \mathbb{Z}_n \).

Notation. If \( R \) is a ring, \( I \subseteq R \) ideal and \( \sim \) the corresponding equivalence relation, then

\[ R/I := R/\sim \text{ the set of equivalence classes.} \]

Note. \( \forall r \in R \)

\[ [rI = \{ r' + I \mid r' - r \in I \}] \]

\[ r' - r = x \]

for some \( x \in I \).
Fix \( \varepsilon > 0 \)
Since \( a_n \to 0 \), \( \exists N_1 \) s.t. \( n > N_1 \) then \( |a_n| < \frac{\varepsilon}{2} \)
Since \( b_n \to 0 \), \( \exists N_2 \) s.t. \( n > N_2 \) then \( |b_n| < \frac{\varepsilon}{2} \)

\[ \Rightarrow \text{for } n > \max(N_1, N_2) \]

\[ |a_n - b_n| = |a_n| + |b_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \]
\[ \Rightarrow (a_n - b_n) \in \mathbb{I} \]

(i) Since \( |c_n| \leq C \), it's bounded: \( \exists C > 0 \) s.t. \( |c_n| < C \)

Given \( \varepsilon > 0 \), \( \exists N \) s.t. \( n > N \) then \( |a_n| < \frac{\varepsilon}{C} \).

Then for \( n > N \),

\[ |a_n \cdot c_n| = |a_n| \cdot |c_n| < \frac{\varepsilon}{C} \cdot C = \varepsilon \]
\[ \Rightarrow a_n \cdot c_n \to 0. \Rightarrow (a_n \cdot c_n) \in \mathbb{I} \]

Exercise: What are all ideals in \( \mathbb{Q} \)?

\( \mathbb{Q}_2 \) and \( \mathbb{Q} \) are ideals.

Suppose \( I \subseteq \mathbb{Q} \) is an ideal, \( I \neq \{0\} \).

Then \( \exists x \in I, x \neq 0. \Rightarrow (1 = x^{-1} \cdot x \in I. \Rightarrow 1 \cdot q \in I \)

\[ \Rightarrow 0 = q \cdot 1 \in I \Rightarrow I = \{0\} \]

Exercise: What are all ideals in \( \mathbb{Z} \)?

Suppose \( I \subseteq \mathbb{Z} \) is an ideal and \( I \neq \{0\} \).

Let \( S = \{ n \in I | n > 0 \} \).

Since \( I \neq \{0\} \), \( \exists x \in I, x \neq 0. \) Then \( 1x = \pm x \in I \)

and \( 1x > 0. \Rightarrow S \neq \emptyset. \)

Let \( n = \min S \) (it exists by well-ordering principle).

Claim: \( I = n \mathbb{Z} \).

Reason: Given \( x \in I \), \( x = q \cdot n + r \), \( 0 \leq r < n \).

Then \( r = x - q \cdot n \in I \)

If \( r > 0 \), then \( r \in S \), which contradicts \( n = \min S \).

\[ \Rightarrow r = 0 \Rightarrow \exists n | x = x \in n \mathbb{Z} \]