Last time:  
- Limits of sequences are unique.

- Cauchy sequences: A sequence \( s_n \) in an ordered field \( F \) [think \( F = \mathbb{Q} \) or \( \mathbb{R} \)] is Cauchy if \( \forall \varepsilon > 0 \exists N \in \mathbb{N} \) s.t. if \( n, m > N \) then \( |s_n - s_m| < \varepsilon \).

- 3.5.4: If a sequence \( s_n \in F \) converges then it is Cauchy.

However \( \forall x \in \mathbb{R} \exists \{a_n\} \subseteq \mathbb{Q} \) with \( a_n \to x \).

Conclusions:  
1. In \( \mathbb{Q} \) there are many Cauchy sequences that don't have a limit in \( \mathbb{Q} \).

2. We should be able to construct \( \mathbb{R} \) out of \( \mathbb{Q} \) as follows: identify \( x \in \mathbb{R} \) with the set of all sequences in \( \mathbb{Q} \) that converge to \( x \).

\[ x \leftrightarrow \{ a_n \} \subseteq \mathbb{Q} \mid a_n \to x. \]

We also saw: \( a_n \to x \), \( b_n \to x \) \( \Rightarrow \) \( a_n - b_n \to 0 \).

We now start our construction of \( \mathbb{R} \).

Observation 18.1: The set of sequences \( \mathbb{Q}^\mathbb{N} \) is a commutative ring \( \mathbb{R} \) is a commutative ring.

Recall: A commutative ring with 1 is:
- A set \( S \) together with two maps \( + : S \times S \to S \), \( \cdot : S \times S \to S \)
- Two distinguished elements \( 0_S, 1_S \in S \), \( 0_S \neq 1_S \)

So that:
1. \( +, \cdot \) are associative and commutative.
2. \( 0_S \) is identity for \( + \), \( 1_S \) is identity for \( \cdot \).
3. \( \forall a \in S \exists (-a) \in S \) s.t. \( a + (-a) = 0_S \)
4. Distributive law holds.

To turn \( \mathbb{Q}^\mathbb{N} = \{ \{a_n\} \} \subset \mathbb{Q} \) into a ring, we define \( 1 \{a_n\} + 3 \{b_n\} = \{a_n + 3b_n\} \), \( \frac{1}{2} \{a_n\} \cdot \{b_n\} = \{\frac{a_n b_n}{2}\} \).
0 is the zero sequence: \( a_n = 0 \) \( \forall n \).

1. In the constant sequence \( a_n = 1 \) \( \forall n \)

\(-1 a_n = (-a_n)\).

It's easy to check \( \mathbb{R}^\mathbb{N} \) is indeed a ring.

Hence, in particular \( \mathbb{Q}^\mathbb{N} \) is set of sequences of rational numbers is a ring.

**Question** Suppose \((\mathbb{R}, +_R, \cdot_R, 0_R, 1_R)\) is a ring and \( S \subseteq \mathbb{R} \) is a subset. Is there a natural way to turn \( S \) into a ring?

\( \exists x \in \mathbb{N} \subseteq \mathbb{Z} \) is not a ring, \( 2\mathbb{Z} \subseteq \mathbb{Z} \) is a ring.

**Lemma 18.2** Suppose \((\mathbb{R}, +_R, \cdot_R, 0_R, 1_R)\) is a ring and \( S \subseteq \mathbb{R} \) a subset such that

1. \( a, b \in S \Rightarrow a - b \in S \)
2. \( a, b \in S \Rightarrow a \cdot b \in S \)
3. \( 1 \in S \)

Then \( S \) is a commutative ring with 1.

**Proof** Since \( 1 \in S \), \( S \neq \emptyset \), \( \forall a \in S \), \( 0 = a - a \in S \).

\( \Rightarrow \forall a \in S \), \( (-a) = 0 - a \in S \) by (1)

\( \Rightarrow \forall a, b \in S \), \( b + a = b - (-a) \in S \) by (1)

\( \Rightarrow \) we can restrict \( +_R : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) to \( S \times S \) and get a map \( +_S : +_R |_{S \times S} : S \times S \to S \)

Since \( +_R \) is associative and commutative, so is \( +_S \).

(2) says: we can restrict \( \cdot_R : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) to \( S \times S \) and get

\( *_S = *_R |_{S \times S} : S \times S \to S \).

\( *_S \) is associative and commutative since \( \cdot_R \) is.

Finally distributive law holds in \( S \) because it holds in \( \mathbb{R} \).
Def: If \( R \) is a ring and \( S \subset R \) is a subset as in 18.2, we say that \( S \) is a subring of \( R \).

Lemma 18.3: The set \( S \) of Cauchy sequences in \( \mathbb{Q} \) is a subring of \( \mathbb{Q}^\infty \).

We need first

Lemma 3.5.7: If \( \{a_n\} \subseteq \mathbb{Q} \) is Cauchy, then \( \exists M \in \mathbb{Q}, M > 0 \), s.t. \( |a_n| \leq M \) for all \( n \in \mathbb{N} \), i.e., \( \{a_n\} \) is bounded.

Proof: Given \( \varepsilon = 1 \), \( \forall N \in \mathbb{N} \) s.t. if \( n, m \geq N \) then

\[ |a_n - a_m| < 1. \]

Let \( M = \max \{|a_1|, |a_N|, |a_{N+1}|\} \).

Then \( |a_k| \leq M \) if \( k \leq N \).

If \( m > N \),

\[ |a_m| = |a_m - a_N + a_N| \leq |a_m - a_N| + |a_N| = |a_N| + 1 \leq M. \]

Proof of 18.3: Clearly the constant sequence \( \{a_n = 1 \} \) is Cauchy, \( \Rightarrow 1 \in S \).

0. If \( \{a_n\}, \{b_n\} \subseteq S \), then \( \forall \varepsilon > 0 \), \( \exists N_1, N_2 \in \mathbb{N} \) s.t.

- if \( n, m \geq N_1 \), then \( |a_n - a_m| < \varepsilon/2 \)
- if \( n, m \geq N_2 \), then \( |b_n - b_m| < \varepsilon/2 \).

Therefore, for \( n, m \geq \max(N_1, N_2) \)

\[ |(a_n - b_n) - (a_m - b_m)| \leq |a_n - a_m| + |b_m - b_n| < \varepsilon/2 + \varepsilon/2, \]

\[ \Rightarrow \{a_n - b_n\} \subseteq S. \]

- Suppose \( \{a_n\}, \{b_n\} \subseteq S \). We argue: \( \{a_n \cdot b_n\} \subseteq S \).

Since \( \{a_n\} \subseteq S \), \( \exists A > 0 \) s.t. \( |a_n| < A \)

Since \( \{b_n\} \subseteq S \), \( \exists B > 0 \) s.t. \( |b_n| < B \).
Now given \( \varepsilon > 0 \), \( \exists N_1 \text{ s.t. if } n, m > N_1 \)
\[ |a_n - a_m| < \frac{\varepsilon}{2B} \]
\( \exists N_2 \text{ s.t. if } n, m > N_2 \)
\[ |b_n - b_m| < \frac{\varepsilon}{2A} \]
Therefore, for \( n, m > \max(N_1, N_2) \)
\[ |a_n b_n - a_m b_m| = |a_n b_n - a_n b_m + a_n b_m - a_m b_m| \]
\[ \leq |a_n| |b_n - b_m| + |b_m| |a_n - a_m| \]
\[ < A \cdot \frac{\varepsilon}{2A} + B \cdot \frac{\varepsilon}{2B} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \]
Now lemma 18.2 \( \Rightarrow (\mathbb{C}, +, \cdot, 0, 1) \) is a
commutative ring with 1.

Ring homomorphisms.

Recall

Def. Let \( R, S \) be two rings. A map \( \varphi : R \rightarrow S \)
\[ \varphi \text{ a homomorphism if it preserves } + \text{ and } \cdot : \]
\[ \forall a, b \in R \quad \varphi(a + b) = \varphi(a) + \varphi(b) \]
\[ \varphi(a \cdot b) = \varphi(a) \cdot \varphi(b) \]

Note: It’s automatic that \( \varphi(0_R) = 0_S, \varphi(-a) = -\varphi(a) \):
Reasons:
(1) \( \varphi(0_R) = \varphi(0_R + 0_R) = \varphi(0_R) + \varphi(0_R) \)
Now add \(-\varphi(0_R)\) to both sides.
\[ \Rightarrow 0_S = \varphi(0_R) \]
(2) \( 0_S = \varphi(0_R) = \varphi(a + (-a)) = \varphi(a) + \varphi(-a) \).
Now add \(-\varphi(a)\) to both sides. Get
\[ -\varphi(a) = -\varphi(a) + \varphi(a) + \varphi(-a) \]
It’s not automatic that \( \varphi(1_R) = 1_S \).
We’ll require it.

Ex. \( \varphi : \mathbb{R} \rightarrow \mathbb{Z} \) is a ring homomorphism, \( \varphi \) \( S \text{ is a subring of } R \text{ } \)
"\text{means: the inclusion map} \)
\[ i : S \rightarrow R, \quad i(s) = s, \text{ is a ring homomorphism.} \]