Last time: A sequence \( (a_n) \) converges to \( l \) if \( \forall \varepsilon > 0 \) \( \exists N \) so that if \( n \geq N \) then \( |a_n - l| < \varepsilon \).

Today: more on limits; Cauchy sequences.

First:

**Fact**: If \( x \in \mathbb{R} \), \( 0 \leq x \) and \( x < \varepsilon \) \( \forall \varepsilon > 0 \) then \( x = 0 \).

**Proof**: If \( x \neq 0 \), then \( x > 0 \). Take \( \varepsilon = x \). Then \( x < \varepsilon \).

Exercise 3.5.5: (was going to be 16.3). Limits are unique if they exist.

**Proof**: Suppose \( a_n \to L_1 \) and \( a_n \to L_2 \).

We argue: \( \forall \varepsilon > 0 \), \( |L_1 - L_2| < \varepsilon \). (Then fact \( \Rightarrow |L_1 - L_2| = 0 \) \( \Rightarrow L_1 = L_2 \).)

Since \( a_n \to L_1 \), \( \exists N_1 \) s.t. \( n \geq N_1 \Rightarrow |a_n - L_1| < \varepsilon/2 \).

Since \( a_n \to L_2 \), \( \exists N_2 \) s.t. \( n \geq N_2 \Rightarrow |a_n - L_2| < \varepsilon/2 \).

Therefore for \( n \geq \max(N_1, N_2) \)

\[
|L_1 - L_2| = |L_1 - a_n + a_n - L_2| 
\leq |L_1 - a_n| + |a_n - L_2| 
< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

Examples of sequences that don't converge:

- \( a_n = n \)
- \( a_n = (-1)^n \)
- \( a_n = (1 - 1)^n \)

Proving this directly from definition requires work.

There is a shortcut:

**Def**: A sequence \( (a_n) \) is **Cauchy** if \( \forall \varepsilon > 0 \exists N \) s.t. \( n, m \geq N \Rightarrow |a_n - a_m| < \varepsilon \).
Exercise 3.5.4: Suppose \( \{a_n\} \) converges \((to L)\).

\( \Rightarrow \) Then \( \{a_n\} \) is Cauchy.

Solution: Since \( a_n \to L \), \( \forall \varepsilon > 0 \), \( \exists N \) st. \( n > N \)

\[ |a_n - L| < \frac{\varepsilon}{2}. \]

Therefore, if \( n, m > N \) then

\[ |a_n - a_m| < |a_n - L + L - a_m| < |a_n - L| + |L - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

\[ \Rightarrow \] \( \{a_n\} \) converges.

\textbf{Ex} \( a_n = n \) is not Cauchy: Let \( \varepsilon = 1 \). Then \( \forall N \in \mathbb{R} \), if \( n > N \), then \( n + 1 > N \), but

\[ |a_{n+1} - a_n| = |n+1 - n| = 1 < 1 = \varepsilon. \]

\( \Rightarrow \) \( \{a_n\} \) does not converge.

\textbf{Ex} \( (-1)^n \) is not Cauchy. Let \( \varepsilon = 1 \). Then \( \forall N \in \mathbb{R} \), if \( n > N \)

\[ |a_{n+1} - a_n| = |(-1)^{n+1} - (-1)^n| = \]

\[ = |(-1)| |(-1) - 1| = 2 \cdot 2 < 1. \]

\( \Rightarrow \) \( a_n = (-1)^n \) does not have a limit; doesn't converge to anything.

More reasons to care about Cauchy sequences:
We've seen: Convergent sequences of real numbers are Cauchy Converge is true as well.

Thm 3.6.14: Any Cauchy sequence of real numbers has a limit (This is really equivalent to existence of sup for sets bounded above)

\( \Rightarrow \) Compare with:
The limit of a convergent sequence of rational numbers need not be rational.
Consider \( \sqrt{2} = 1.41421356237 \ldots \)

Now let \( a_1 = 1, \ a_2 = 1.4, \ a_3 = 1.41, \ a_4 = 1.414 \ldots \)
\( a_n \to \sqrt{2} \)

In fact:

**Lemma 17.1**  \( \forall x \in \mathbb{R} \) \exists sequence \( \{a_n\}_{n=1}^\infty \) s.t.
\[
\begin{align*}
& a_n \in \mathbb{Q} \forall n \ni n \in \mathbb{N} \\
& a_n \to x.
\end{align*}
\]

**Proof** \( \forall n \in \mathbb{N} \exists r_n \in \mathbb{Q} \) s.t. \( x - r_n < r_n < x \) by 3.2.6.
\( \Rightarrow \) (1) \( x - r_n > 0 \)
\( \Rightarrow \) (2) \( x - r_n < \frac{1}{n} \).
\( \Rightarrow \) \( 1 - r_n \) \( < \frac{1}{n} \).
\( \Rightarrow \) \( x - r_n \) \( < \frac{1}{n} \).

Now, \( \forall \varepsilon > 0, \exists N \in \mathbb{N} \) s.t. \( \frac{1}{N} < \varepsilon \).

Then if \( n \geq N \)
\[
| x - r_n | < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.
\]
\( \Rightarrow \) \( \{r_n\} \) converges to \( x \).

\[\square\]

**Note:** The sequence \( \{r_n\} \) we constructed is very far from being unique: \( \frac{1}{n} \to 0 \) and \( \frac{1}{n^2} \to 0 \ldots \)

**Key idea of section 3.5:** Construct \( \mathbb{R} \) as a set of equivalence classes of Cauchy sequences of rational numbers.

**What's the equivalence relation?**

**Lemma 17.2** Suppose \( a_n \to x \) and \( b_n \to x \). Then \( a_n - b_n \to 0 \).

**Proof** Need to show: \( \forall \varepsilon > 0 \) \( \exists N \in \mathbb{N} \) s.t. if \( n \geq N \) then
\[
| (a_n - b_n) - 0 | = | a_n - b_n | < \varepsilon.
\]
Since \( a_n \to x \), \( \exists N_1 \) s.t. if \( n \geq N_1 \) then \( |a_n - x| < \varepsilon/2 \).
Since \( b_n \to x \), \( \exists N_2 \) s.t. if \( n \geq N_2 \) then \( |b_n - x| < \varepsilon/2 \).
For $n \geq N = \max(N_1, N_2)$

$$|a_n - b_n| = |a_n - x + x - b_n| \leq |a_n - x| + |x - b_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

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**Strategy for constructing $\mathbb{R}$:**

1. Let $\mathbb{B} = \{a_n \mid n \in \mathbb{N}\}$ such that $\{a_n\}$ is Cauchy.

2. Make $\mathbb{B}$ into a commutative ring with unity.

3. Let $I = \{a_n \mid a_n \text{ converges to } 0\}$

   Define ~ on $\mathbb{B}$ by

   $$(a_n \sim b_n) \iff (a_n - b_n) \in I.$$

   Check that ~ is an equivalence relation and that $\mathbb{B}/\sim$ is an ordered field with least upper bound property.