1a Prove that $|a| \leq |a - b| + |b|$ for any $a, b \in \mathbb{R}$. Hint: triangle inequality.
b Prove that for any $\epsilon > 0$ and any $x \in \mathbb{R}$

$$(x < \epsilon \text{ and } -x < \epsilon) \Rightarrow |x| < \epsilon.$$
c Prove that

$$||a| - |b|| \leq |a - b|.$$  

Hint: parts (a) and (b) may be useful.

2 In class we defined a sequence $\{a_n\}$ to converge to $L$ if and only if for any $\epsilon > 0$ there is $N \in \mathbb{R}$ so that for any natural number $n$ if $n \geq N$ then $|a_n - L| < \epsilon$.

Prove that the definition above is equivalent to: "$\forall \epsilon > 0 \exists K \in \mathbb{N}$ so that for any $n \in \mathbb{N}$ with $n > K$ we have $|a_n - L| < \epsilon$.”

3 Prove that if $\{a_n\}$ is a sequence of real numbers that converges to $L > 0$ then there is $n_0 \in \mathbb{N}$ so that $a_n > 0$ for all $n \geq n_0$. Hint: let $\epsilon = L/2$...

4 Let $\{a_n\}$ be a constant sequence of real numbers: $a_n = c$ for some $c \in \mathbb{R}$ and all $n$. Prove that $\{a_n\}$ converges to $c$.

5 Suppose $\{a_n\}$ is a sequence of real numbers, $L \in \mathbb{R}$ and $|a_n - L| < \frac{1}{n}$ for all $n \in \mathbb{N}$. Prove that $\{a_n\}$ converges to $L$.

6 Suppose $A \subset \mathbb{R}$ is bounded above and $L = \sup A$. Prove that there is a sequence $\{a_n\}$ with $a_n \in A$ and $\lim a_n = L$. Hint: Show that for any $n \in \mathbb{N}$ the number $L - \frac{1}{n}$ is not an upper bound of $A$. Therefore there is $a_n \in A$ with $L - \frac{1}{n} < a_n$. Now use problem 5.

7 Recall that a natural number $p > 1$ is prime if (and only if) the only natural numbers that divide it are 1 and $p$.

a Prove that $p$ is prime if and only if

$$\gcd(p, n) = 1 \text{ or } p$$

for any $n \in \mathbb{Z}$, $n \neq 0$. Hint: this should be easy.
b Prove that if $p$ is prime, $n \in \mathbb{Z}$ and $p$ does not divide $n$ then there are $a, b \in \mathbb{Z}$ so that $ap + bn = 1$. Hint: Homework 5, problem 9.
c Prove that if $p$ is prime and $\bar{n} \in \mathbb{Z}_p$ is not $\bar{0}$ then there is $\bar{b} \in \mathbb{Z}_p$ so that

$$\bar{b}\bar{n} = \bar{1}.$$  

Hint: (b).
d Prove that if $m > 1$ is a natural number that is not prime then $\mathbb{Z}_m$ has zero divisors.
e Prove that $\mathbb{Z}_n$ is a field if and only if $n$ is prime.