Homework 5, Math 347 , Prof. Eugene Lerman
Due Wednesday, February 25, 2015 (in class)

1 Prove that the set \( \mathbb{Q} \) of rational numbers is countable. Hint: prove that \( \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \) is countable.

2 If \( n \in \mathbb{N} \) and \( A_1, \ldots, A_n \) are sets, their Cartesian product \( A_1 \times \ldots \times A_n \) is defined to be the set of all \( n \) tuples \((a_1, \ldots, a_n)\) with \( a_i \in A_i \) for all \( i \):
\[
A_1 \times \cdots \times A_n := \{(a_1, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, \ldots, n\}
\]
Prove that if the sets \( A_1, \ldots, A_n \) are all countable then so is their product \( A_1 \times \cdots \times A_n \).

3 Given two sets \( X \) and \( Y \) denote the set of all functions from \( X \) to \( Y \) by \( \text{Map}(X, Y) \). The same set is also denoted by \( Y^X \).

Given three (nonempty) sets \( A, B \) and \( C \) show that there is a bijection \( \text{Map}(A \times B, C) \rightarrow \text{Map}(A, \text{Map}(B, C)) \).

Hint: if \( f : A \times B \rightarrow C \) is a function then for all \( a \in A \) we have a function \( f_a : B \rightarrow C \); it is defined by
\[
f_a(b) := f(a, b) \quad \text{for all } b \in B.
\]

4 Prove that if \( A \) is uncountable and \( A \subset B \) then \( B \) is uncountable.

5 Prove that the set \( \text{Map}(\mathbb{N}, \mathbb{N}) \) of all functions from \( \mathbb{N} \) to itself is not countable. Hint: look at the proof that \( \mathbb{R} \) is not countable.

The next four problems deal with divisibility of integers. In these exercises you may not assume that an integer can be factored into primes.

6 Suppose \( a, b \) are two nonzero integers. Prove that if \( a \mid b \) and \( b \mid a \) then either \( a = b \) or \( a = -b \).

7 Suppose a nonzero integer \( a \) divides \( b \in \mathbb{Z} \) and divides \( c \in \mathbb{Z} \). Prove that \( a \) divides \( nb + mc \) for all \( n, m \in \mathbb{Z} \).

8 An natural number \( d \in \mathbb{N} \) is a greatest common divisor (gcd) of \( a, b \in \mathbb{Z} \) if and only if
\[
(1) \ d \ \text{divides} \ a \text{ and } b \\
(2) \ \text{if } d' | a \text{ and } d' | b \text{ then } d' | d.
\]
Prove that if \( d_1, d_2 \) are two gcd’s of \( a \) and \( b \) then \( d_1 = d_2 \), that is, gcd’s are unique.

9 Prove that any two nonzero integers \( a \) and \( b \) have a gcd. Moreover the gcd has to be of the form \( na + mb \) for some \( n, m \in \mathbb{Z} \).

Hints: Consider
\[
W = \{xa + yb \mid x, y \in \mathbb{Z}, xa + yb > 0\}
\]
By well-ordering \( W \) has the smallest element, call it \( d \). Use division algorithm to prove that \( d | a \) and \( d | b \); you know that \( a = qd + r \) with \( 0 \leq r < d \); if \( r \neq 0 \) there should be a contradiction. Next prove that this \( d \) is, in fact, the gcd of \( a \) and \( b \).