1. Let $V$ and $W$ be two vector spaces over a field $F$ and $T : V \to W$ a linear map. Prove that if $T$ is a bijection then its inverse $T^{-1} : W \to V$ is linear as well.

Hints: you need to show that $T^{-1}$ preserves addition and multiplication by scalars. To show that $T^{-1}$ preserves addition, consider $w_1, w_2 \in W$. Show that $w_1 = T(v_1), w_2 = T(v_2)$ for some $v_1, v_2 \in V$. Now compute $T^{-1}(w_1 + w_2) = T^{-1}(T(v_1) + T(v_2)) = ...$

2. Let $T : V \to W$ be linear map between two vector spaces over a field $F$. The kernel of $T$ is, by Definition 2.2.11, the set

$$\ker T := \{v \in V \mid T(v) = 0\}.$$

(a.) Prove that $\ker T$ is a subspace of $V$ (see Definition 2.1.25).

(b.) Show that $T$ is 1-1 if and only if $\ker T = \{0\}$.

3. Let $F$ be a field, $a \in F$.

(a.) Prove that map $\varphi : F[x] \to F$ given by

$$\varphi(p) := p(a)$$

is a homomorphism. The homomorphism $\varphi$ is called the evaluation at $a$.

(b.) Prove that $\ker \varphi$ is the ideal $(x - a)$ consisting of all multiples of the polynomial $x - a$. Hint: division algorithm and/or one of its corollaries.

(c.) Prove that $F[x]/(x - a)$ is isomorphic to $F$.

4. Let $F$ be a field and $I \subset F[x]$ an ideal. Prove that there is a polynomial $f \in F[x]$ so that $I = (f)$. That is, prove that all elements of $I$ are multiples of a single polynomial $f$. Hints: what is $f$ if $I = 0$? $I = F[x]$?

Now assume $I \subset F[x]$ is proper. Consider

$$W = \{\deg p \mid p \in I, p \neq 0\}.$$

Argue that $W$ has the smallest element and pick $f \in I$ so that $\deg f = \min W$. Now argue as in the case of ideals in $\mathbb{Z}$.

5. Let $F$ be a field. Consider the map $T : F[x] \to F[x]$ defined by

$$T(a_0 + a_1 x + \cdots + a_n x^n) = a_1 + 2a_2 x + \cdots + na_{n-1}x^{n-1}.$$  

(If $F = \mathbb{R}$ then $T$ is the map that sends a polynomial to its derivative).

(a.) Prove that $T$ is linear.

(b.) Is $T$ injective? Prove your answer.

(c.) Prove that for $F = \mathbb{Q}$ the map $T$ is onto. Is $T$ onto if $F = \mathbb{Z}_2$? Prove your answer.