Homework 8, Math 347, **Prof. Eugene Lerman**
Due Friday, March 28, 2008

A sequence $\{a_n\}$ is **monotone** if it is either non-increasing or non-decreasing. Recall that monotone bounded sequences always converge.

1 Let $\{a_n\}$ and $\{b_n\}$ be two monotone sequences. Is it true that $c_n = a_n + b_n$ is monotone? Prove or give a counterexample.

2 Give an example of a sequence of real numbers so that it is

(a) Cauchy but not monotone;

(b) monotone but not Cauchy;

(c) bounded but not Cauchy.

3 Let $\{a_n\}$ be a convergent sequence of real numbers with $a_n \geq 0$ for all n.
(a) Prove that if $\lim_{n\to\infty} a_n = 0$ then $\lim_{n\to\infty} \sqrt{a_n} = 0$ as well.
(b) Prove that if $\lim_{n\to\infty} a_n = L > 0$ then $\lim_{n\to\infty} a_n = \sqrt{L}$.
Hint: for all $x, y > 0$

$$|\sqrt{x} - \sqrt{y}| = \frac{|\sqrt{x} - \sqrt{y}||\sqrt{x} + \sqrt{y}|}{|\sqrt{x} + \sqrt{y}|} = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq \frac{|x - y|}{\sqrt{y}}$$

4 Let $\{a_n\}$ be a sequence defined recursively by $a_1 = 1$, $a_{n+1} = \sqrt{a_n + 1}$ for all $n \geq 1$.
(a) Prove (by induction): $a_n < 2$ for all n and $a_n < a_{n+1}$ for all n. Conclude that $\lim_{n\to\infty} a_n$ exists (What theorem are you using?).
(b) Use the equation $a_{n+1} = \sqrt{a_n + 1}$ to show that $L^2 = L + 1$. What is L?
Hint: Previous problem may be useful for computing $\lim_{n\to\infty} \sqrt{a_n + 1}$.

5 Prove that a sequence of complex numbers $\{z_n\}$ converges to $L \in \mathbb{C}$ if and only if ($Re(z_n) \to Re(L)$ and $Im(z_n) \to Im(L)$) .