1 Definition: A function \( f : \mathbb{C} \to \mathbb{R} \) is continuous at \( L \in \mathbb{C} \) if for every \( \epsilon > 0 \) there is \( \delta > 0 \) so that
\[
|z - L| < \delta \Rightarrow |f(z) - f(L)| < \epsilon.
\]
Prove, using the definition above, that function \( f(z) = |z| \) is continuous at all points of \( \mathbb{C} \).

2 Prove that if \( f : \mathbb{C} \to \mathbb{R} \) is continuous and \( \{z_n\} \subset \mathbb{C} \) converges to \( L \) then \( \{f(z_n)\} \) converges to \( f(L) \).

3 Prove that if a function \( f : \mathbb{R} \to \mathbb{R} \) is constant, then it is continuous.

4 Prove that if \( f, g : \mathbb{R} \to \mathbb{R} \) are continuous (at all points of \( \mathbb{R} \)) then so are their sum \( f + g \) and product \( fg \).

Hint: Let \( \{x_n\} \) be a sequence converging to \( L \). Then \( f(x_n) \to f(L) \), \( g(x_n) \to g(L) \) and \( f(x_n) + g(x_n) \to ... \)

5 Find the mistake in the “proof” of the following bogus Claim any continuous function \( f : (0, 1) \to \mathbb{R} \) is bounded.

**Proof:** Suppose \( f \) is not bounded. Then for any \( n \in \mathbb{N} \) there is \( x_n \in (0, 1) \) with \( |f(x_n)| \geq n \). Since \( 0 < x_n < 1 \), \( \{x_n\} \) is bounded. Hence it has a convergent subsequence \( \{x_{n_k}\} \). Let \( L = \lim_{k \to \infty} x_{n_k} \). Since \( f \) is continuous \( f(x_{n_k}) \to f(L) \). On the other hand \( |f(x_{n_k})| \geq n_k \) by construction. Hence \( \{f(x_{n_k})\} \) is unbounded and cannot converge. Contradiction.

Hint: \( f(x) = \frac{1}{x} \) is continuous on \( (0, 1) \) but is not bounded.