Worksheet for the final, Math 347, Monday, April 28, 2008 **Prof. Eugene Lerman**

1 Let $p, g \in \mathbb{R}[x]$ be two polynomials with $\deg p > \deg g$ and $p|g$. Prove that $g = 0$. Hint: suppose $g \neq 0$....

2 Suppose X is a set, $\sim$ an equivalence relation and $X/\sim$ is the set of the equivalence classes.

(a) Prove that the function $f : X \to X/\sim$, which takes $x \in X$ to its equivalence class, is onto. Hint: it's very short.

(b) Prove that for any $x, y \in X$

$$x \sim y \quad \Leftrightarrow \quad [x] = [y].$$

3 Prove that $f(x) = 3x$ defines a bijection from $\mathbb{R}$ to $\mathbb{R}$. Does it define a bijection from $\mathbb{Z}$ to $\mathbb{Z}$? If not, what goes wrong? If it does, what's the inverse?

4 (a) Let A and B be two sets. Show that if A and B are countable then so are $A \cup B$ and $A \times B$.

(b) Suppose A is not countable and $A \subset C$. Prove that C is not countable.

5 Prove that if $f, g : \mathbb{R} \to \mathbb{R}$ are continuous, then so is their composition $f(g(x))$. Hint: Fix $x_0 \in \mathbb{R}$. Suppose $x_n \to x_0$....

6 Prove that if p is prime then $\sqrt[3]{p}$ is irrational.
Hint: suppose $\sqrt[3]{p} = \frac{r}{s}$ where $r, s \in \mathbb{N}$ are relatively prime. Then $s^3 p = r^3$ and so $p|r^3$...

7 Suppose $f : A \to B$ and $g : B \to A$ are two functions so that $f(g(b)) = b$ for all $b \in B$. Prove that f is onto and g is 1-1. Give an example where f is not 1-1 and g is not onto.

8 Which of the two statements below are true? Explain.

$$(\forall a \in \mathbb{R})\,(\exists p(x) \in \mathbb{R}[x])\,(p(a) = 0)$$

$$(\exists a \in \mathbb{R})\,(\forall p(x) \in \mathbb{R}[x])\,(p(a) = 0)$$

9 Prove that if $\{a_n\}$ is a sequence of real numbers with

$$|a_{n+1} - a_n| < \frac{1}{3^n}$$

for all $n \in \mathbb{N}$, then $\{a_n\}$ is Cauchy, hence converges.

10 Is the set

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^6 + y^4 = 1\}$$

the graph of a function from $\mathbb{R}$ to $\mathbb{R}$? Explain.

11 Find the greatest common divisor of 6188 and 4709 and write it as a linear combination of 6188 and 4709.

12 Prove that $11^{2007} - 1$ is divisible by 10. Hint: in $\mathbb{Z}/10$, $[11^{2007}] = \dots$

13 Let c be a real number and $f : \mathbb{R} \to \mathbb{R}$ a continuous function. Prove, **using only the** $\varepsilon - \delta$ **definition of continuity** that cf is continuous. Hint: you may wish to consider two cases: $c = 0$ and $c \neq 0$.

14 Find and prove a formula for the sum :

$$1 + 5 + 9 + \cdots + (4n + 1).$$

15 Prove that the limit of a sequence $\{a_n\}$ of real numbers is unique.

16 For what values of $z \in \mathbb{C}$ is the series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ guaranteed to converge?

17 Prove that for any $n \in \mathbb{N}$ and any prime p if $p|(a_1 \dots a_n)$ then there is i, $1 \leq i \leq n$ so that $p|a_i$. Hints: Induction. What is the base case? What is the inductive step?

18 Prove that $gcd(x, n) = 1$ if and only if there is $y \in \mathbb{Z}$ so that $[x][y] = [1]$ in $\mathbb{Z}/n$.