MATH 347 Homework #11 Due 4/25/08 (in class)

Note Title

#1 a) Prove that if $f: A \to B$ and $g: C \to D$ are onto, then so is $h: A \times C \to B \times D$, $h(a,c) = (f(a), g(c))$

Since f is onto, $\forall b \in B$ $\exists a \in A$ with $f(a) = b$. Since g is onto, $\forall d \in D$ $\exists c \in C$ so that $g(c) = d$. Therefore, for any $(b,d) \in B \times D$ $\exists (a,c) \in A \times C$ so that $(b,d) = (f(a), g(c)) = h(a,c)$.

b) Prove that if $f: A \to B$ and $g: C \to D$ are 1-1, then so is $h: A \times C \to B \times D$, $h(a,c) = (f(a), g(c))$.

Suppose $h(a,c) = h(a',c')$. Then $(f(a), g(c)) = (f(a'), g(c')) \Rightarrow$ $f(a) = f(a')$ and $g(c) = g(c')$. Since f & g are 1-1, $a = a'$ and $c = c'$. $\Rightarrow$ $(a,c) = (a',c')$.

#2 Prove that inverses are unique: if $f: A \to B$ is a map and $g, h: B \to A$ are two maps so that $g(f(a)) = a = h(f(a))$ $\forall a \in A$ and and $f(g(b)) = b = f(h(b))$ for all $b \in B$ then $h = g$.

Since $g(f(a)) = a$ for all $a \in A$, $g(f(h(b))) = h(b)$ for all $b \in B$. On the other hand, since $f(h(b)) = b$ $\forall b \in B$, $g(f(h(b))) = g(b)$ $\forall b \in B$

$\Rightarrow$ $h(b) = g(f(h(b))) = g(b)$ $\forall b \in B$

#3 a) Let $f: A \to B$ be a bijection and let $f^{-1}: B \to A$ be its inverse. Prove that f^{-1} is a bijection. What is the inverse of f^{-1}?

Recall A map g is a bijection $\Leftrightarrow$ g has an inverse. Now,

$$\left.\begin{array}{l} f(f^{-1}(b)) = b \quad \forall b \in B \\ \text{and} \quad f^{-1}(f(a)) = a \quad \forall a \in A. \end{array}\right\} \Rightarrow f \text{ is an inverse of } f^{-1}!$$

$\Rightarrow$ f^{-1} is a bijection.

b) Prove that if $f: A \to B$ and $g: B \to C$ are invertible, then so is $g \circ f: A \to C$. Prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

For any $a \in A$, $(f^{-1} \circ g^{-1}) \circ (g \circ f)(a) = f^{-1}(g^{-1}(g(f(a)))) = f^{-1}(f(a)) = a$.

Similarly, for any $c \in C$, $(g \circ f) \circ (f^{-1} \circ g^{-1})(c) = g(f(f^{-1}(g^{-1}(c)))) = g(g^{-1}(c)) = c$

$\Rightarrow$ $f^{-1} \circ g^{-1}$ is an inverse of $g \circ f$. $\Rightarrow$ $g \circ f$ is invertible.

#4 Prove that the set of irrational numbers is not countable.

Claim If A and B are countable then so is $A \cup B$.

Proof of claim: Since A and B are countable, there are surjective maps $f: \mathbb{N} \to A$ and $g: \mathbb{N} \to B$. Define $h: \mathbb{N} \to A \cup B$ by $h(2n) = f(n)$, $h(2n+1) = g(n)$ for all $n \in \mathbb{N}$. It's easy to check that h is onto. $\square$

If the set B of all irrational were countable, then $\mathbb{Q} \cup B$ would be countable by Claim above. But $B \cup \mathbb{Q} = \mathbb{R}$, which is not countable.

$\Rightarrow$ the set of irrational numbers is not countable.

#5 Is the map $\varphi: \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$, $\varphi(p,q) = p/q$ 1-1? Is it onto? Prove your answers.

Any rational number x is of the form p/q, where $p, q \in \mathbb{Z}$ and $q \neq 0$

We may always assume $q > 0$: if $q < 0$, $x = \frac{-p}{-q}$ and $-q > 0$.

Therefore any rational number x equals p/q for some $p \in \mathbb{Z}$, $q \in \mathbb{N}$; i.e. $x = \varphi(p,q)$ for some $(p,q) \in \mathbb{Z} \times \mathbb{N}$. $\Rightarrow$ φ is onto.

φ is not 1-1: $\varphi(1,1) = \varphi(2,2)$.

#6 Is the map $f: (0,\infty) \to \mathbb{R}$, $f(x) = \ln x$ 1-1? Onto? Prove your answers.

The map $f(x) = \ln x$ has an inverse $g(y) = e^y$.

Hence $\ln x$ is 1-1 and onto.