Cauchy sequences converge.

Note Title

<u>Theorem</u> Let $\{a_n\}$ be a Cauchy sequence in $\mathbb{R}$. Then $\{a_n\}$ converges.

<u>Proof</u> Since $\{a_n\}$ is Cauchy, it is bounded. Therefore we may define

$$t_n = \sup\{a_k \mid k \geq n\}$$
$$b_n = \inf\{a_k \mid k \geq n\}.$$

<u>Note:</u>

1) $b_1 \leq b_n \leq a_n \leq t_n < t_1$
2) t_n is decreasing
3) b_n is increasing

Since $\{b_n\}$ is increasing and bounded above, $\lim b_n$ exists. Call it b.
Since $\{t_n\}$ is decreasing and is bounded below, $\lim t_n$ exists. Call it t.
Since $b_n \leq t_n$ for all n, $b \leq t$.
We now argue that $t = b$. For then, by the squeezing theorem, $\{a_n\}$ converges to $t = b$.

We argue by contradiction (this is where the assumption that $\{a_n\}$ is Cauchy will be used).

$t_n \geq t$ for all n. In fact $t = \inf\{t_n \mid n \in \mathbb{N}\}$.
Hence, for any $\varepsilon > 0$, $t_n > t - \varepsilon$ (all n).

Now, for any set S, $\sup S > \beta \Rightarrow \exists\, s \in S$ with $s > \beta$
(Quick proof: if $\alpha = \sup S > \beta$, then $\beta < \frac{\alpha+\beta}{2} < \alpha$. Since $\frac{\alpha+\beta}{2} < \alpha$, it cannot be an upper bound of S. So $\exists\, s \in S$ with $s > \frac{\alpha+\beta}{2} > \beta$.)

Hence $\exists\, a_k \in \{a_j \mid j \geq n\}$ with $a_k > t - \varepsilon$.

That is, $\forall \varepsilon > 0$, $\forall n \in \mathbb{N}$ $\exists\, k \geq n$ with $a_k > t - \varepsilon$.

Similarly, $b_n \leq b \Rightarrow \forall \varepsilon > 0$ $\forall n \in \mathbb{N}$, $b_n < b + \varepsilon$

$b_n = \inf\{a_j \mid j \geq n\} < b + \varepsilon \Rightarrow \exists\, \ell \geq n$ so that $a_\ell < b + \varepsilon$

Now suppose $b \neq t$. Then $b < t$. Let $\varepsilon = \frac{1}{3}(t - b)$

Since $\{a_n\}$ is Cauchy, $\exists N$ s.t.

$$n, m \geq N \Rightarrow |a_n - a_m| < \varepsilon$$

On the other hand, $\exists\, k, \ell \geq N$ so that

$$a_k > t - \varepsilon \quad \text{and} \quad a_\ell < b + \varepsilon.$$

But then $a_k - a_\ell > (t - \varepsilon) - (b + \varepsilon) = (t - b) - 2\varepsilon = \varepsilon$.

Contradiction. Therefore $b = t$ and the squeezing theorem applies.