The exam will cover lectures 15 – 30. On the algebra side you need to be comfortable with rings, fields, integral domains, ordered rings, homomorphisms, ideals, quotient rings, polynomial rings, the division algorithm for polynomials, the first isomorphism theorem, characteristic of a ring.

On the analysis side you need to know about convergence of sequences, Cauchy sequences, lower and upper bounds of subset of an ordered field, least upper bound property of \( \mathbb{R} \), triangle inequality for real and complex numbers, sequences and series of complex numbers, absolute convergence of series, comparison and ratio tests.

1. Let \( \{z_n\} \) be a sequence of complex numbers. Suppose \( z_n \to L \). Prove that
   (a) \( \{z_n\} \) is bounded;
   (b) \( \{z_n\} \) is Cauchy.

2. Does every Cauchy sequence of complex numbers have a limit? Explain.

3. Is the field of complex numbers \( \mathbb{C} \) an ordered field? Prove your answer. We used order to define absolute value of real numbers. How is absolute value of complex numbers defined?

4. Let \( \{z_n\} \) be a sequence of complex numbers. What does it mean to say that the series \( \sum z_n \) converges?

5. Find all real numbers \( x \) such that the series \( \sum \frac{x^n}{n!} \) converges. Caution: the ratio test tells you nothing if the limit of ratios is 1; so you need to use something else...

6. Let \( \{z_n\}, \{w_n\} \) be two convergent sequences in \( \mathbb{C} \). Prove that their sum \( \{z_n+w_n\} \) converges to \( \lim z_n + \lim w_n \).

7. Let \( \{z_n\} \) be a constant sequence: there is some \( c \in \mathbb{C} \) so that \( z_n = c \) for all \( n \). Prove that \( \{z_n\} \) converges to \( c \).

8. Let \( \{z_n\} \) be a sequence of complex numbers. Suppose that the series \( \sum |z_n| \) converges. Prove that \( \sum z_n \) converges.

9. Prove that a sequence of complex numbers \( \{z_n\} \) converges to \( L \) if and only if \( (Re z_n \to Re L \) and \( Im z_n \to Im L \).

10. Let \( \mathcal{C} \) denote the set of Cauchy sequences of real numbers.
    (a) Prove that \( \mathcal{C} \) is a subring of the ring \( \mathbb{R}^\mathbb{N} \) of all sequences of real numbers (addition and multiplication in \( \mathbb{R}^\mathbb{N} \) is defined term by term).
    (b) Prove that the map \( \varphi : \mathcal{C} \to \mathbb{R} \) given by
        \[
        \varphi(\{a_n\}) := \lim a_n
        \]
    is a ring homomorphism.
    (c) Prove that \( \varphi \) above is surjective.
    (d) What is the kernel of \( \varphi \)?
    (e) What does the first isomorphism theorem tell you in this case?

11. Suppose \( F \) is an ordered field and \( R \subset F \) is a subring. Prove that \( R \) can be ordered so that the inclusion \( \iota : R \to F \) is order-preserving.

12. Let \( F \) be a field. Suppose the homomorphism \( \varphi : \mathbb{Z} \to F \) given by \( \varphi(n) = n1_F \) is injective. Prove that \( F \) contains a subfield isomorphic to \( \mathbb{Q} \).

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(13) Let $F$ be a field and the homomorphism $\varphi : \mathbb{Z} \to F$ given by $\varphi(n) = n1_F$ as in the problem above. Now assume that the image $\varphi(\mathbb{Z})$ is isomorphic to $\mathbb{Z}_n$.
   (a) What must be the kernel of $\varphi$?
   (b) Use the fact that $F$ is a field to prove that $n$ must be a prime number.

(14) Let $R$ be a commutative ring. An ideal $I \subset R$ is proper if $I \neq 0$ or $R$.
   (a) Let $I$ be a proper ideal of $R$. What equivalence class is the zero in the quotient ring $R/I$? If $R$ has a unity $1_R$ does $R/I$ have a unity? If yes, what is it?
   (b) What is the kernel of the canonical homomorphism $\pi : R \to R/I$?

(15) Suppose $\{a_n\}$ is a monotonically increasing sequence of real numbers which is bounded above. Prove that $\{a_n\}$ converges to $\sup\{a_n \mid n \in \mathbb{N}\}$.

(16) Suppose $\{a_n\}$ is a bounded sequence of real numbers (not necessarily convergent). Then for each $n \in \mathbb{N}$
   
   \[ b_n := \inf \{a_k \mid k \geq n\} \]
   
   is a real number. Prove that the sequence $\{b_n\}$ converges. Hint: problem 15 above may be useful. What do you know about the sequence $\{b_n\}$?

(17) Let $F$ be a field, $a \in F$.
   (a) Prove that map $\varphi : F[x] \to F$ given by
      
      \[ \varphi(p) := p(a) \]
      
      is a homomorphism. The homomorphism $\varphi$ is called the evaluation at $a$.
   (b) Prove that $\ker \varphi$ is the ideal $\langle x - a \rangle$ consisting of all multiples of the polynomial $x - a$.
      Hint: division algorithm and/or one of its corollaries.
   (c) Prove that $F[x]/\langle x - a \rangle$ is isomorphic to $F$.

(18) (This may be a somewhat harder problem.) Let $F$ be a field and $I \subset F[x]$ an ideal. Prove that there is a polynomial $f \in F[x]$ so that $I = \langle f \rangle$. That is, prove that all elements of $I$ are multiples of a single polynomial $f$. Hints: what is $f$ if $I = 0$? $I = F[x]$?
   Now assume $I \subset F[x]$ is proper. Consider
   
   \[ W = \{\deg p \mid p \in I, p \neq 0\} \]
   
   Argue that $W$ has the smallest element and pick $f \in I$ so that $\deg f = \min W$. Now argue as in the case of ideals in $\mathbb{Z}$.

(19) Prove that $\mathbb{Z}_{10}/[5]\mathbb{Z}_{10}$ is isomorphic to $\mathbb{Z}_5$.
   Hint: We have two canonical surjective homomorphisms $f : \mathbb{Z} \to \mathbb{Z}_{10}$ and $g : \mathbb{Z}_{10} \to \mathbb{Z}_{10}/[5]\mathbb{Z}_{10}$. Compute the kernel of $g \circ f : \mathbb{Z} \to \mathbb{Z}_{10}/[5]\mathbb{Z}_{10}$.

(20) Let $R$ be a commutative ring and $I \subset R$ an ideal. Prove that any two equivalence classes $r + I, r' + I \in R/I$ have the same cardinality. Hint: consider the map $f : r + I \to R$ given by $f(r + i) = r' + (r - r') + i$. What is the image of $f$? Is $f$ 1-1?

(21) Let $R, R'$ be two rings (with 1’s as usual). Suppose $\varphi : R \to \mathbb{R}'$ is a homomorphisms. Do not assume that $\varphi(1_R) = 1_{R'}$. Let $e = \varphi(1_R)$
   (a) Prove that $e^2 = e$.
   (b) Assume next that $R'$ is an integral domain and $e \neq 0$. Prove that $e = 1_{R'}$.  

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