Ordered pairs.

Given two (nonempty) sets $A$ and $B$ we would like to define their Cartesian product $A \times B$.

It's the set of ordered pairs

$$A \times B = \{ (a, b) \mid a \in A, b \in B \}.$$ 

Q. So what's an ordered pair $(a, b)$?

A. $(a, b)$ is a set with the following property:

for all $a' \in A, b' \in B$

$$(a, b) = (a', b') \iff (a = a' \text{ and } b = b').$$

There are several ways to construct $(a, b)$.

For example, we could set

$$(*) \quad (a, b) = \{ (a, b), (b, a) \}.$$ 

(See Wikipedia for a proof that $(a, b)$ defined by $(*)$ does have the required property).

There are other constructions. In practice what matters is not the construction but the property.

**Ex.** $A = \{ 0, 1 \}, B = \{ 1, 2, 3 \}$

$A \times B = \{ (0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3) \}$

$B \times A = \{ (1, 0), (2, 0), (3, 0), (1, 1), (2, 1), (3, 1) \}$

**Note**

$A \times B \neq B \times A$ (as sets)

**Ex."** $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, the set of ordered pairs of real numbers.

**Ex.""** For any set $A$, $\emptyset \times A = \emptyset$ since $\emptyset \times A$ has
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no elements. Similarly $A \times \emptyset = \emptyset$.

**Definition 2.1** A relation on a set $X$ is a subset $R$ of $X \times X$.

**Example** (parity relation on $\mathbb{Z}$)

$$R = \{(n, m) \in \mathbb{Z} \times \mathbb{Z} \mid \text{n and m are both odd or }\text{n and m are both even}$$

$$= \{(n, m) \in \mathbb{Z} \times \mathbb{Z} \mid n - m \text{ is even} \}$$

**Example** Let $X$ be any set. The equality relation on the set $X$ is

$$\Delta = \{(x, y) \in X \times X \mid x = y \}$$

**Example** $X = \mathbb{R}$.

$$\mathcal{R} = \{(x, y) \in \mathbb{R} \mid x < y \}$$ the "less than" relation.

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**Equivalence relations**

**Definition** (see Salt, 1.6.1) An equivalence relation on a set $X$ is a relation $R \subseteq X \times X$ such that

1. **(ER1) Reflexivity** For any $a \in X$, $(a, a) \in R$
2. **(ER2) Symmetry** If $(a, b) \in R$ then $(b, a) \in R$
3. **(ER3) Transitivity** If $(a, b)$ and $(b, c)$ are in $R$ then $(a, c) \in R$. 


Example 2.3

Recall: \( n \in \mathbb{Z} \) is even if \( n = 2k \) for some \( k \in \mathbb{Z} \).

Claim:
\[
\mathcal{R} = \{(n,m) : \mathbb{Z} \times \mathbb{Z} \mid n-m \text{ is even}\}
\]

is an equivalence relation.

Proof:
(i) Since \( 0 = 2 \cdot 0 \), \( 0 \) is even,
\[
\Rightarrow \forall n \in \mathbb{Z} \text{ (i.e., for any } n \in \mathbb{Z} \text{) } n-n = 0 \text{ is even}
\Rightarrow (n,n) \in \mathcal{R}.
\]
(ii) If \( (n,m) \in \mathcal{R} \), then \( n-m = 2k \) for some \( k \in \mathbb{Z} \)
\[
\Rightarrow m-n = -(n-m) = -2k = 2(-k)
\Rightarrow (m,n) \in \mathcal{R}.
\]
(iii) Suppose \( (n,m), (m,r) \in \mathcal{R} \).
Then \( n-m = 2k \) and \( m-r = 2l \) for some \( k, l \in \mathbb{Z} \).
\[
\Rightarrow n-r = (n-m) + (m-r) = 2k + 2l = 2(k+l)
\Rightarrow (n,r) \in \mathcal{R}.
\]

Remark: With this notation the definition of an equivalence relation \( \mathcal{R} \) on a set \( X \) translates into:

(i) \( \forall a \in X \) \( a \sim a \)
(ii) \( \forall a, b \in X \) if \( a \sim b \) then \( b \sim a \)
(iii) \( \forall a, b, c \in X \) if \( a \sim b \) and \( b \sim c \) then \( a \sim c \).
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Functions

Informally, a function \( f \) from a set \( A \) to a set \( B \) assigns to each element \( a \in A \) an element \( f(a) \in B \).

We write: \( f : A \rightarrow B \)

\( A \) is the domain of the function \( f \), \( B \) is the range of \( f \)

(\( B \) is also called the codomain and the target of \( f \))

Formally these are two ways to proceed:

1) "function" is a primitive notion, just like "set."

2) Everything is a set. So a function \( f : A \rightarrow B \) should be a set of some sorts.

   Now given \( f : A \rightarrow B \) we have the set

   \[ \text{graph}(f) = \{(a,b) : a \in A, b = f(a)\} \]

   which is a subset of \( A \times B \).

Consequently

Def. (Sally, 1.7.3) Let \( A \) and \( B \) be sets. A function from \( A \) to \( B \) is a subset \( R \subseteq A \times B \) such that

i) \( \forall a \in A \) \( \exists b \in B \) with \( (a,b) \in R \)

ii) if \( (a,b) \) and \( (a,b') \) \( \in R \) then \( a = a' \)

("each element of \( A \) occurs exactly once as \( a \) first coordinate ")

Ex. \( A = \emptyset \), \( B \) is a set

\( \emptyset \times B = \emptyset \).

And \( \emptyset \subseteq \emptyset \) is a function. Why?

Q. Is there a function from \( B \neq \emptyset \) to \( \emptyset \)?
Most mathematicians (most of the time) don't think of functions as graphs. We think of them as rules that assign elements to elements.

**Example** For every set $A$ we have the identity function $\text{id}_A : A \to A$, $\text{id}_A(a) = a$.

( $\text{graph (id}_A) = \{(a, b) \in A \times A \mid b = a\}$

**Note:** if $A = \emptyset$, $\text{graph (id}_A) = \emptyset$.

**Functions can be composed:**

Given two functions $f : A \to B$, $g : B \to C$.

Their composite $g \circ f$ is defined by

$$(g \circ f)(a) = g(f(a)).$$

**Example** If $g(x) = \sin(x)$ and $f(x) = x^2$,

$$(g \circ f)(x) = \sin(x^2).$$

**Theorem** (Sally, 1.7.3) Composition of functions is associative: for any three functions $f : A \to B$, $g : B \to C$ and $h : C \to D$,

$$(h \circ (g \circ f)) = (h \circ g) \circ f.$$  

**Proof** For any $a \in A$,

$$((h \circ (g \circ f))(a) = (h \circ (g \circ f))(a) = h(g(f(a))) = h(g(f(a))) = h((g \circ f)(a)) = (h \circ (g \circ f))(a).$$

Since $a \in A$ is arbitrary,

$$(h \circ (g \circ f)) = (h \circ g) \circ f.$$  

For all triples of composable functions. $\blacksquare$