Last time defined Cauchy sequences:

- \( \{a_n\} \) is Cauchy if \( \forall \varepsilon > 0 \ \exists N \ \text{st.} \ \forall n, m \geq N \Rightarrow |a_n - a_m| < \varepsilon \).

Proved:
- If \( \{a_n\} \) converges then \( \{a_n\} \) is Cauchy.
- \( \forall x \in \mathbb{R} \exists \text{ sequence } \{a_n\}, \ a_n \to \mathbb{R} \ \text{st. } a_n \to x \)

Hence: In \( \mathbb{R} \) there are Cauchy sequences with no limit in \( \mathbb{R} \).

We'll see later: In \( \mathbb{R} \) any Cauchy sequence converges.

We proved:
- If \( \{a_n\}, \{b_n\} \) are two sequences in \( \mathbb{R} \) with
  \( \lim a_n = L = \lim b_n \) then \( \{a_n - b_n\}\) converges to 0.

We plan to define \( \mathbb{R} \) as:

\[ \mathbb{R} = \left\{ \{a_n\} \mid \{a_n\} \text{ a Cauchy sequence in } \mathbb{Q} \right\} \]

where \( \{a_n\} \sim \{b_n\} \iff a_n - b_n \to 0 \).

Lemma 18.1. Let \( X \) be a set and \( R \) a commutative ring (with 1).

Then \( \text{Map}(X, R) = \) the set of all functions from \( X \) to \( R \)

is a commutative ring with 1.

Sketch of proof:

- We define \( 0 \in \text{Map}(X, R) \) to be the function \( x \mapsto 0 \in R \), \( \forall x \in X \).

- We define \( 1 \in \text{Map}(X, R) \) to be the constant function \( x \mapsto 1 \in R \), \( \forall x \in X \).

- We define \( + \) on \( \text{Map}(X, R) \) by:
  \[ (f + g)(x) = f(x) + g(x) \quad \forall x \in X \]

- We define \( \cdot \) on \( \text{Map}(X, R) \) by:
  \[ (f \cdot g)(x) = f(x) \cdot g(x) \quad \forall x \in X \]

Exercise: \( +, \cdot \) as defined above are commutative and associative,

\[ (f + g) \cdot h = (f \cdot h) + (g \cdot h) \]

\( \forall f, g, h \in \text{Map}(X, R) \).
The additive inverse of $f : X \to R$ is $-f$ defined by
\[ (-f)(x) = -f(x) \quad \forall x \in X \]
$1 \cdot f = f = f \cdot 1$ for all $f : X \to R$.

Recall, a subset $S$ of a ring $R$ is a subring if $S \neq \emptyset$ and:
- $s_1, s_2 \in S \Rightarrow s_1 + s_2, s_1 \cdot s_2 \in S$
- $0_R \in S$
- $(S, +, \cdot; 0_R)$ is a ring

Example: $R$ a ring, $X$ a set, $S = \{ f : X \to R \mid f$ is constant $\}$ is a subring of $\text{Map}(X, R)$.

Definition (homomorphism) let $K, L$ be two rings and $f : K \to L$ a function. $f$ is a homomorphism if $f$ preserves $+$ and $\cdot$:
\[ f(a + b) = f(a) + f(b) \]
\[ f(a \cdot b) = f(a) \cdot f(b) \]
for all $a, b \in K$.

$\text{Ex}$: $\pi : \mathbb{Z} \to \mathbb{Z}_n$, $\pi(a) = [a]$

in a homomorphism since:
\[ \pi(a + b) = [a + b] = [a + b] = \pi(a) + \pi(b) \]
\[ \pi(a \cdot b) = [a \cdot b] = [a] \cdot [b] = \pi(a) \cdot \pi(b) \]

$\text{Ex}$: $X$ a set, $R$ a ring,
$\varphi : R \to \text{Map}(X, R)$
$\varphi(r) = \text{constant function } x \mapsto r \quad \forall x$
$\varphi$ is a homomorphism.
Lemma 18.2: If $f: K \rightarrow L$ is a homomorphism then

(i) $f(0_K) = 0_L$
(ii) $f(-a) = -(f(a))$ for all $a \in K$.

Proof:
(i) $f(0_K) = f(0_K + 0_K) = f(0_K) + f(0_K)$
Now add $-(f(0_K))$ to both sides. We set
$0_L = f(0_K)$.

(ii) For any $a \in K$,
$f(0_K) = f(a + (-a)) = f(a) + f(-a)$.
Add $-(f(a))$ to both sides. We get
$-(f(a)) = f(-a)$.

Note: If $f: K \rightarrow L$ is a homomorphism, it's not automatic that $f(1_K) = 1_L$.

Example:
$K = IR$, $L = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. $\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \mid a, b \in IR \right\}$: diagonal $2 \times 2$ matrices.

$f: K \rightarrow L$, $f(a) = \begin{pmatrix} a \\ 0 \end{pmatrix}$ is a homomorphism

$f(a+b) = \begin{pmatrix} a+b \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} b \\ 0 \end{pmatrix}$

$f(ab) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} \cdot \begin{pmatrix} b \\ 0 \end{pmatrix}$

But $f(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq 1_L = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Definition: A homomorphism $f: K \rightarrow L$ between two rings is an isomorphism if there is a homomorphism $g: L \rightarrow K$ so that $g \circ f = \text{id}_K$, $f \circ g = \text{id}_L$.

Note: If $f$ is an isomorphism then $f$ is 1-1 and onto.
Lemma 18.4. If \( f : K \to L \) is a homomorphism that is 1-1 and onto, then \( f^{-1} : L \to K \) is a homomorphism.

Warning: Many textbooks define homomorphisms as bijective homomorphisms.

Example: \( S = \{ \frac{n}{m} \in \mathbb{Z} \mid n \in \mathbb{Z}, m \in \mathbb{Z}, m \neq 0 \} \) is a subring.

\[
\begin{align*}
\frac{n}{1} + \frac{m}{1} &= \frac{n \cdot 1 + m \cdot 1}{1 \cdot 1} = \frac{nm}{1} \\
\frac{n}{1} \cdot \frac{m}{1} &= \frac{n \cdot m}{1 \cdot 1} = \frac{nm}{1} \\
\frac{0}{1} &= 0
\end{align*}
\]

and \( -\frac{n}{1} = \frac{-n}{1} \in S \).

\[ f : \mathbb{Z} \to S, \quad f(n) = \frac{n}{1} \text{ is an homomorphism:} \]
\[ f(n + m) = \frac{n + m}{1} = \frac{n}{1} + \frac{m}{1} = f(n) + f(m) \]
\[ f(n \cdot m) = \frac{n \cdot m}{1} = \frac{n}{1} \cdot \frac{m}{1} = f(n) \cdot f(m) \]
\[ f \text{ is onto. And } f(n) = f(n') \Rightarrow \frac{n}{1} = \frac{n'}{1} \Rightarrow n \cdot 1 = n' \cdot 1 \Rightarrow n = n'. \]

So \( f \) is 1-1.

Fact: \( f \) is an homomorphism.

We think \( S \) is a "copy of \( \mathbb{Z} \)" inside \( \mathbb{Q} \).

Proof of 18.3. Let \( \varphi = f^{-1} \). We need to show: \( \forall a, b \in L \)

\[ \varphi(a + b) = \varphi(a) + \varphi(b), \quad \varphi(ab) = \varphi(a) \cdot \varphi(b) \]

\[ f(\varphi(a + b)) = a + b = f(\varphi(a)) + f(\varphi(b)) = f(\varphi(a) + \varphi(b)) \]

Since \( f \) is a homomorphism,

\[ \varphi(a + b) = \varphi(a) + \varphi(b) \] (since \( f \) is 1-1)

Similarly,

\[ f(\varphi(ab)) = ab = f(\varphi(a)) \cdot f(\varphi(b)) = f(\varphi(a) \cdot \varphi(b)) \]

\[ \varphi(ab) = \varphi(a) \cdot \varphi(b) \]  \( \square \)