13. Countable and Uncountable Sets

Last time: Discussed the axiom of choice. Prove that the axiom of choice is equivalent to:

“Any surjective map \(g : B \to A\) has a right inverse. That is, if there is a surjective map \(g : B \to A\) then there is a map \(f : A \to B\) with \(g \circ f = \text{id}_A\).”

Defined a set \(A\) to be countable if either \(A\) is finite or if there is a bijection \(h : \mathbb{N} \to A\).

Didn’t have time to prove:

Lemma 12.7 A nonempty set \(A\) is countable if and only if there is a surjective map \(\mathbb{N} \to A\).

Proof. ( \(\implies\) ) If \(A\) is finite, there exists a bijection \(f : \{1, \ldots, n\} \to A\) for some \(n \in \mathbb{N}\). Define \(g : \mathbb{N} \to A\) by

\[
g(i) = \begin{cases} f(i) & \text{if } 1 \leq i \leq n \\ f(1) & \text{if } i > n \end{cases}.
\]

If \(A\) is infinite, there is a bijection \(g : \mathbb{N} \to A\), which is onto.

( \(\impliedby\) ). Suppose \(g : \mathbb{N} \to A\) is onto. If the set \(A\) is finite, we’re done. Suppose \(A\) is infinite. We need to construct a bijective map \(\mathbb{N} \to A\), or, equivalently a bijective map \(A \to \mathbb{N}\).

By Theorem 12.3, we have an injection \(f : A \to \mathbb{N}\). Let \(D = f(A)\). Then \(f : A \to D\) is a bijection. By Lemma 12.5, we have a bijection \(h : D \to \mathbb{N}\). The map \(h \circ f : A \to \mathbb{N}\) is the desired bijection. \(\Box\)

Corollary 13.1. If \(A\) is countable and \(h : A \to B\) is onto, then \(B\) is countable.

Proof. By Lemma 12.7 there is a surjective map \(f : \mathbb{N} \to A\). The composite \(h \circ f : \mathbb{N} \to \mathbb{N}\) is surjective since it is the composition of two surjective maps. By Lemma 12.7 the existence of a surjective map from \(\mathbb{N}\) to \(B\) implies that \(B\) is countable. \(\Box\)

Uncountable sets are easy to come up with.

Example 13.2. By Cantor’s theorem there is no surjective map from the set \(\mathbb{N}\) of natural numbers to its power set \(\mathcal{P}(\mathbb{N})\). Therefore \(\mathcal{P}(\mathbb{N})\) is uncountable.

Theorem 13.3 (also due to Cantor). The set of real numbers \(\mathbb{R}\) is uncountable.

Remark 13.4. A positive real number \(x\) has a unique decimal expansion

\[
x = a_n a_{n-1} \ldots a_1 a_{-1} a_{-2} \ldots\quad a_i \in \{0, \ldots, 9\}
\]

as long as (as long as we don’t allow the expansion to end in an infinite sequence of 9’s). Recall why this is the case. First of all for any \(q \in \mathbb{R}\) with \(|q| < 1\),

\[
\sum_{n=1}^\infty q^n = \frac{q}{1 - q}.
\]

Therefore

\[
0.999999 \ldots = 9 \cdot \sum_{n=1}^\infty \left(\frac{1}{10}\right)^n = 9 \cdot \frac{\frac{1}{10}}{1 - \frac{1}{10}} = 1.
\]

Consequently \(a_n \ldots a_1 a_{-1} a_{-2} \ldots a_{-k}9999999 \ldots = a_n \ldots a_1 a_{-1} a_{-2} \ldots a_{-k}000000 \ldots + 10^{-k}\).

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Proof. We will argue that the set $(0, \infty)$ of positive real numbers is uncountable. This is enough since the exponential map

$$\exp : \mathbb{R} \to (0, \infty), \quad \exp(x) = e^x$$

is a bijection.

Suppose the set $(0, \infty)$ of positive real numbers is countable. That is, suppose there is a surjective function $x : \mathbb{N} \to (0, \infty)$, $n \mapsto x_n$. Consider the decimal expansions of $x_n$'s:

$$x_1 = a_1^{(1)} \ldots a_1^{(1)} a_{-1}^{(1)} \ldots$$
$$x_2 = a_2^{(2)} \ldots a_2^{(2)} a_{-2}^{(2)} \ldots$$
$$\vdots$$

We produce a positive real number $r$ that’s not on the list above as follows. Choose digits $r_{-1}, r_{-2}, \ldots, r_{-n} \in \{1, \ldots, 9\}$ such that

$$r_{-1} \neq a_{-1}^{(1)}$$
$$r_{-2} \neq a_{-2}^{(2)}$$
$$\vdots$$
$$r_{-n} \neq a_{-n}^{(n)}$$
$$\vdots$$

and let $r = 0.r_{-1}r_{-2}\ldots r_{-n} \ldots$. For example if our list starts with

$$a_1 = 102.130\ldots$$
$$a_2 = 10,029.1460\ldots$$
$$a_3 = 0.1390\ldots$$
$$\vdots$$

we may choose $r = 0.251\ldots$.

Note that $r \neq x_n$ for any $n$. This is because $r_{-n} \neq a_{-n}^{(n)} =$ $n^\text{th}$ digit of $x_n$ past the decimal point. Contradiction! Hence there is no surjective map $\mathbb{N} \to \mathbb{R}$. In other words $\mathbb{R}$ is uncountable. \qed}

Back to countable sets.

**Lemma 13.5.** The product $\mathbb{N} \times \mathbb{N}$ is countable.

**Sketch of proof.** The easiest thing to do is to draw a picture:
A counting of elements of $\mathbb{N} \times \mathbb{N}$.

But pictures may be deceiving, so here is a formula for a map $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$:

\[ f(k, n - k + 1) = 1 + 2 + \cdots + (n - 1) + k \]

for all $n \in \mathbb{N}$, and all $k \in \mathbb{N}$ with $1 \leq k \leq n$.

Then

\[
\begin{align*}
  f(1, 1) &= f(1, 1 - 1 + 1) = (1 - 1) + 1 = 1 \\
  f(1, 2) &= f(1, 2 - 1 + 1) = (2 - 1) + 1 = 2 \\
  f(2, 1) &= f(2, 2 - 2 + 1) = (2 - 1) + 2 = 3 \\
  f(1, 3) &= f(1, 3 - 1 + 1) = 1 + (3 - 1) + 1 = 4 \\
  &\vdots
\end{align*}
\]

Check that $f$ is a bijection. \hfill \Box

**Lemma 13.6.** If two sets $A$ and $B$ are countable then so is their product $A \times B$.

**Proof.** Since $A$ and $B$ are countable, we have surjections

\[ f : \mathbb{N} \to A, \quad g : \mathbb{N} \to B. \]

Then

\[ (f, g) : \mathbb{N} \times \mathbb{N} \to A \times B, \quad (f, g)(i,j) = (f(i), g(j)) \]

is onto (check this!). Since $\mathbb{N} \times \mathbb{N}$ is countable, the product $A \times B$ is countable by Corollary 13.1. \hfill \Box

**Lemma 13.7.** Let $\{A\}_{n=1}^{\infty}$ be a countable family of countable sets. Then the union $\bigcup_{n=0}^{\infty} A_n$ is countable.

**Proof.** Since each $A_n$ is countable we have a surjection $f_n : \mathbb{N} \to A_n$. Now define $\varphi : \mathbb{N} \times \mathbb{N} \to \bigcup_{n=0}^{\infty} A_n$ as follows:

\[ \varphi(k, n) := f_n(k). \]
I claim that the map \( \varphi \) is onto. Let’s check that. If \( a \in \bigcup_{n=0}^{\infty} A_n \), \( a \in A_n \) for some \( n \). Since \( f_n : \mathbb{N} \to A_n \) is onto there is \( k \in \mathbb{N} \) such that \( a = f_n(k) \). Hence \( a = \varphi(k, n) \). Since \( \mathbb{N} \times \mathbb{N} \) is countable, the union \( \bigcup_{n=0}^{\infty} A_n \) is countable by Corollary 13.1. □